On some stochastic control problems with state constraints

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Contents

Abbreviations

DPP	Dynamic Programming Principle
HJ	Hamilton-Jacobi
HJB	Hamilton-Jacobi-Bellman
USC	upper semi-continuous
LSC	lower semi-continuous

Notation

$\mathbb{R}^n$	the Euclidean n-dimensional space
$\mathbb{R}^{n\times m}$	space of $(n \times m)$ real matrices
$\mathcal{S}^n$	space of symmetric matrices in $\mathbb{R}^{n\times n}$
Ω	space of the realizations ω
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
$\mathcal{B}(\cdot)$	Brownian motion
$\{\mathcal{F}_t\}_{t\geq 0}$	filtration
$\{\mathbb{F}_t\}_{t\geq 0}$	filtration generated by the Brownian motion
$B(x, r)$	ball of center x and radius r
$d_D(\cdot)$	Euclidean distance function to the set D (with sign)
$d_D^+(\cdot)$	positive Euclidean distance function to the set D
$USC(D)$	the set of upper semi-continuous functions defined on D
$LSC(D)$	set of lower semi-continuous functions defined on D
$\mathcal{J}^{2,+}, \mathcal{J}^{2,-}$	semijets
$\mathcal{P}^{1,2,+}, \mathcal{P}^{1,2,-}$	parabolic semijets
$D\varphi$	gradient
$D^2\varphi$	Hessian matrix

Chapter 1

General introduction and main contribution

1.1 General introduction

The purpose of this thesis is to study some control problems in presence of state constraints via the Hamilton-Jacobi-Bellman approach.

The usual objective of *control theory* is to influence the behavior of a system in order to attain some desired goals: minimize/maximize a cost, reach a target, stabilize the system, etc. For a given controlled system, governed by ordinary or stochastic differential equations, it is natural, for modeling purposes, to restrict the state space taking into account the presence of *state constraints*. In this case, the controller is allowed to act only by means of those control inputs that satisfy such constraints. These controls are called the *admissible controls*.
The aim of the present work is to analyze some stochastic control problem from the theoretical and computational point of view and to use the tools of *optimal control* theory to establish a general framework for dealing with the presence of state constraints.

Optimal control is a branch of the control theory strictly related with optimization. For this kind of problems the aim is to find a control strategy such that a certain optimality criterion is achieved. This criterion is usually expressed by a cost, that is a functional depending on the choice of the control input.

The systematic study of deterministic optimal control problems received a significant improvement starting from the 50's, strongly motivated from the growing interest at that time for aerospace engineering. The field of stochastic optimal control was developed since the 70's for applications related to finance. In 1971 Robert C. Merton [142] was the first to use stochastic control for studying portfolio optimization for "risky" and "riskless" assets. A fundamental contribution in this field was also the introduction of the financial model formulated by Fischer Black and Myron Scholes in [44].

Two main approaches can be found in literature for dealing with optimal control problems: the Pontryagin Maximum Principle approach and the Dynamic Programming approach. The Pontryagin Maximum Principle, formulated in 1956 by the Russian mathematician Lev Semenovich Pontryagin and his school [152], provides a general set of necessary conditions for the optimality of a strategy and it tapes its roots in the method of Lagrange multipliers applied in constrained optimization.

For the results provided in this thesis, we will follow the Dynamic Programming approach developed in the 50's by Richard Bellman [40]. In this method a central role is played by the *value function*, that is the optimal value of the optimization problem. At the basis of the approach there is the Bellman's principle of Optimality:

> An optimal policy has the property that no matter what the previous decision (i.e., controls) have been, the remaining decisions must constitute an optimal policy with regard to the state resulting from those previous decisions.
> (Bellman, *Dynamic Programming*, 1957)

This principle essentially contains the information on the procedure to follow for recursively solving a complex optimal control problem by its decomposition in simpler subproblems.
In mathematical language, the property of the optimal control problem described by the Bellman's principle is expressed by the fact that the value function satisfies a particular functional inequality, the *Dynamic Programming Principle*.

Starting from this functional equation and under suitable regularity assumptions, it can be proved that the value function satisfies a special kind of nonlinear partial differential equation called the Hamilton-Jacobi-Bellman (HJB) equation. A first order equation is typically obtained for optimal control problems set in a deterministic framework, whereas second order equations arise in the stochastic case. The high nonlinearity of the problem requires to consider weak notions of solutions. A breakthrough in this direction was the introduction in the early 80's of the notion of *viscosity solution* by Michael G. Crandall and Pierre-Louis Lions [86]. In [132] and [133] Lions characterized the value function associated with optimal control problems for controlled diffusion processes as the unique continuous viscosity solution of a second order HJB equation. It turned out that viscosity solutions theory is the suitable framework for providing existence and uniqueness for a wider class of nonlinear equations: the Hamilton-Jacobi (HJ) equations. This is also a very convenient context for analyzing the numerical methods related to such nonlinear equations. In the later years a wide literature was produces on the subject providing different existence and uniqueness results (see [110, 83, 113, 109, 111, 116]). When state constraints are taken into account the characterization of the value function as the viscosity solution of an HJB equation becomes much more complicated and, in absence of further assumption, uniqueness cannot in general be proved. We mention the works of Soner [160, 161], Frankowska et al. [101, 100], Katsolulakis [117], Barles and Rouy [35], Bouchard and Nutz [62] and many others for the discussion of the suitable sets of assumptions to be considered.

This thesis has the objective of exploiting the technical tools coming from viscosity theory for HJB equations in order to develop new ideas for facing the presence of state constraints. We can summarize the philosophy of the main contribution of the thesis as follows: once the state constrained control problem is stated in its most generality, a more or less classical *unconstrained* optimal control problem is introduced. For a suitable definition of such a problem, the issues related to the presence of state constraints are in some sense eluded and the theoretical analysis is developed around this new "auxiliary" unconstrained optimal control problem. In a significant part of the thesis, represented essentially by Chapters 3 and 5, this goal is attained making use of the existing link between optimal control and reachability. On one hand, state constrained reachability problems are connected to the solution of unconstrained optimal control problems. On the other hand any state constrained optimal control problem is translated in a reachability one establishing, in this way, a sort of equivalence result between state constrained and unconstrained optimal control theory. In Chapter 4, generalizing the so-called Zubov method, the controllability properties of a stochastic system under state constraints are investigated by the introduction of a convenient unconstrained optimal control problem. Extending the arguments presented in Chapter 3, optimal control theory can also be used to analyze the viability properties of a given domain. This will be applied, in Chapter 6, for studying a very particular class of ergodic optimal control problems.

1.2 Main contributions and perspectives

Along all the thesis, for a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a Brownian motion $\mathcal{B}(\cdot)$, the dynamics we will consider is a system of controlled stochastic differential equations (SDE) of the following form

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))d\mathcal{B}(s) \\ X(t) = x, \end{cases}$$

where $t \geq 0$ and u denotes the control. For any choice of the control u and the initial time and space (t, x), we will denote by $X^u_{t,x}(\cdot)$ the unique solution of the previous equation. In some case the initial time $t = 0$ is fixed and the corresponding trajectory is simply denoted by $X^u_x(\cdot)$.

1.2.1 Optimal control theory and backward reachability

A first important contribution of the thesis is to deeply investigate the link between optimal control problems and *backward reachability*. This will be strongly used in Chapters 3 and 5.
A stochastic reachability problem is a particular control problem where the controller aims to steer the state of the system towards a given target $\mathcal{T}$. Given a fixed time horizon $T \geq 0$, this requirement can be expressed by the following relation:

$$X^u_{t,x}(T) \in \mathcal{T} \quad \text{for some choice of the control } u.$$

Analogously, for a given set $\mathcal{K}$, a state constrained requirement can be enunciated by:

$$X^u_{t,x}(s) \in \mathcal{K}, \ \forall s \in [t, T].$$

Working in a stochastic framework it is necessary to specify in which sense the previous condition has to be satisfied (with non-zero probability? With probability one? With probability grater then a certain value?). In this part of the thesis the requirements are considered in the strongest sense, that is with probability one or, as we will say, almost surely (a.s.).
Our research aims to establish a sort of duality result between the following two kinds of problems:

Minimize a cost $J(t,x,u)$ over all the admissible trajectories $X^u_{t,x,}(\cdot)$ of the system;	Find the starting point x : $X^u_{t,x}(T) \in \mathcal{T}$ and $X^u_{t,x}(s) \in \mathcal{K}, \ \forall s \in [t, T]$.

In the deterministic case this kind of result is nowadays well known. It has been shown in [2] and [79] that the epigraph of the value function associated with finite horizon ($T < \infty$) optimal control problems can be seen as a backward reachable set for which the target is represented by the epigraph of the terminal cost function. In [16] and [78] similar results have been obtained respectively for infinite horizon and minimal time problems, obtaining a description of the epigraph of the associated value functions by means of the viability properties of the system. We point out that, using the typical vocabulary coming from viability theory, the concept of state constrained backward reachability is given by the notion of viability kernel and capture basin [11]. On the other hand a generalization of the so-called *level set* approach [147] has been used in [47, 121, 122, 98] for characterizing the backward reachable set as a level set of the value function associated with a suitable optimal control problem.

The formulation of optimal control problems by means of reachability notions has fundamental advantages in the treatment of state constraints. As shortly mentioned in the previous general introduction, when state constraints are taken into account, the application of the Dynamic Programming techniques requires that some additional "compatibility assumptions" between the dynamics and the set of state constraints are satisfied. These assumptions are necessary for proving uniqueness results for the HJB equation associated to state constrained problems and they have the role of supplying the lack of information on the boundary of the state constraints due to the fact that here only the super-solution property holds.
For what concerns the deterministic framework, their earliest introduction goes back to the works of Soner [160] and [161] where appeared for the first time the so-called "inward pointing conditions". Formulated for an autonomous deterministic dynamics, i.e. $\sigma \equiv 0$ and $b(t, x, u) = b(x, u)$, the Soner's conditions can be stated as follows:

$$\inf_{u \in U} \mathbf{n}(x) \cdot b(x, u) < 0 \qquad \text{for any } x \in \partial\mathcal{K},$$

where U is the set of control values and $\mathbf{n}$ denotes the exterior normal vector. From the geometrical point of view this condition says that at each point on the boundary of the state constraints there exists at least one vector field that points inside the constraint. Later extension and discussion on more or less restrictive sets of assumptions can be found for instance in [77, 112, 101, 100, 168, 136, 143]. In the stochastic case a first extension of the controllability conditions was proposed by Katsoulakis in [117]. Other contributions are [35, 114, 62] (more references will be given in Chapter 5). Roughly speaking, in the stochastic setting the typical idea is to "complete" the Soner-type condition with some degeneracy assumption on the diffusion term σ.
However, also remaining in the deterministic context, this kind of conditions may fail even for very simple cases, as the following example aims to show.

Example 1.2.1. Consider a one dimensional mechanical system governed by the following Newton's law

$$\ddot{x} = f(x, \dot{x}, u).$$

With the usual change of variables $y_1 = x$ and $y_2 = \dot{x}$, the dynamics can be rewritten as a 2-dimensional first order system with respect to the new state $y \equiv (y_1, y_2)$ and one has

$$\dot{y} = b(y, u) := \begin{pmatrix} y_2 \\ f(y_1, y_2, u) \end{pmatrix}.$$

Any boundedness requirement on the state and the velocity of the form $x \in [-M, M]$ and $\dot{x} \in [-R, R]$, is expressed by the following state constraints on the variable y:

$$y \in \mathcal{K} := [-M, M] \times [-R, R].$$

In this case, at each boundary point such that $y_1 = M$ (similar arguments hold for $y_1 = -M$) one has

$$\mathbf{n}(M, y_2) \cdot b((M, y_2), u) = y_2$$

for any control $u \in U$ and any $y_2 \in (-R, R)$. Hence, the inward pointing condition fails as soon as $y_2 \geq 0$. In particular when $y_1 = \pm M$ the inner product above does not depend at all on the control, so every assumption of this form would fail.

The reformulation of optimal control problems by means of reachability has the advantage of allowing to deal with state constraints even when the compatibility conditions are not satisfied. In the deterministic case, this fact has been well pointed out in [2, 16, 47, 78, 79].
In [16, 78, 79] once established that the epigraph of the value function is a suitable viability kernel, its values are computed by using the tools developed in [158]. In [2] the backward reachability problem characterizing the epigraph of the value function is solved by a level set method and the state constraints are managed without any further assumption by an exact penalization technique.

The contribution of this thesis is to extend this ideas to the stochastic setting.
In Chapter 3 we will discuss the application of the level set approach for solving state constrained reachability problems in the stochastic case. We will show in Proposition 3.3.2 and Remark 3.3.4 that a state constrained backward reachable set can be characterized as the zero level set of two value functions associated with two different optimal control problems: a classical optimal control problem with integral cost and an optimal control problem with maximum cost. Motivated by the promising results obtained for deterministic systems in [2] and [47], we provide a complete analysis for the second one taking into account the general class of optimal control problems associated with the cost

$$J(t, x, u) = \mathbb{E}\left[\psi\left(X^u_{t,x}(T), \max_{s \in [t,T]} g(X^u_{t,x}(s))\right)\right].$$

In this case, as already pointed out in [29, 37], the application of the dynamic programming techniques requires the introduction of an auxiliary variable $y \in \mathbb{R}$, so that the optimal control problem object of our study will be

$$\vartheta(t, x, y) = \inf_{u \in \mathcal{U}} \mathbb{E}\left[\psi\left(X^u_{t,x}(T), \max_{s \in [t,T]} g(X^u_{t,x}(s)) \vee y\right)\right],$$

where $a \vee b := \max(a, b)$. This leads us to the study of second order HJB equations with oblique derivative boundary conditions of the following form

$$\begin{cases} -\partial_t \vartheta + H(t, x, D\vartheta, D^2\vartheta) = 0 & t \in [0, T), y > g(x) \\ -\partial_y \vartheta = 0 & t \in [0, T), y = g(x) \\ \vartheta(T, x, y) = \psi(x, y) & y \geq g(x). \end{cases}$$

By Theorems 3.5.2 and 3.5.4, ϑ is characterized as the unique viscosity solution of such equation. The numerical aspects are investigated. A general algorithm is described and a convergence result proved, generalizing in this way the results obtained in [30] for the particular case $g(x) = |x|$. Another important content of this chapter is the presentation of error estimates for a semi-Lagrangian scheme. Considered at a first stage only the discretization in time given by $h = T/N$, the scheme is recursively defined by

$$\begin{aligned} &V(t_N, x, y) = \psi(x, g(x) \vee y) \\ &V(t_{n-1}, x, y) = \Psi[V](t_n, x, y \vee g(x)) \quad \text{for } n = N, \dots, 1 \end{aligned}$$

where, denoted by p the dimension of the Brownian motion and by $\{\sigma_1, \dots, \sigma_p\}$ the column vectors of the diffusion matrix σ, the operator Ψ is given by

$$\Psi[V](t, x, y) := \min_{u \in U} \frac{1}{2p} \sum_{k=1}^{2p} V\big(t, x + hb(t, x, u) + \sqrt{hp}(-1)^k \sigma_{\lfloor \frac{k+1}{2} \rfloor}(t, x, u), y\big).$$

The error estimates for the fully discrete scheme presented in Theorem 3.7.6 represent a first attempt to apply the technique of the "shaking coefficients" introduced by Krylov in [119] and [120] (see also the works of Barles and Jakobsen [31, 32, 33]) to HJB equations with derivative boundary conditions. Finally, in the last part of the chapter numerical tests are proposed and the methods discussed are applied for solving state constrained backward reachability problems by the level set approach.
In Chapter 5 the reversed link is investigated. We start from a state constrained optimal control problem

$$\begin{aligned} v(t, x) = \inf_{u \in \mathcal{U}} \Bigg\{ \mathbb{E}\Bigg[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s)) ds \Bigg] : \\ X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t, T] \text{ a.s.} \Bigg\}. \end{aligned}$$

Applying the techniques described by Bouchard and Dang in [58] for the unconstrained case, the epigraph of the value function v is characterized by means of the following state constrained reachability problem in an augmented state and control space:

$$\begin{aligned} \text{Find } (x, z): \quad & (X^u_{t,x}(T), Z^{u,\alpha}_{t,x,z}(T)) \in \text{Epigraph}(\psi) \\ & \text{and} \\ & X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t, T] \quad \text{a.s.} \\ & \text{for some control } u \in \mathcal{U}, \alpha \in \mathcal{A}, \end{aligned}$$

for a suitable definition of the process $Z^{u,\alpha}_{t,x,z}(\cdot)$. At this point the study developed in Chapter 3 can be used and in Theorem 5.4.3 the epigraph of the value function v is characterized as the zero level set of a function

$$w(t, x, z) = \inf_{(u,\alpha) \in \mathcal{U} \times \mathcal{A}} \mathbb{E}\Bigg[\max(\psi(X^u_{t,x}(T)) - Z^{\alpha,u}_{t,x,z}(T), 0) + \int_t^T g_{\mathcal{K}}(X^u_{t,x}(s)) ds \Bigg].$$

The advantage of this procedure is that now the dynamic programming techniques are only applied on this "auxiliary" unconstrained optimal control problem. The technical difficulties of the chapter arise from the unboundedness of the controls $\alpha \in \mathcal{A}$. This will result in the unboundedness of the Hamiltonian associated with w and in the necessity of finding a new well posed formulation of the HJB equation. Developing the techniques used in [64], Theorem 5.5.6, together with the comparison principle stated by Theorem

5.6.1, finally characterizes the value function w as the unique viscosity solution of the following generalized HJB equation

$$\begin{cases} \sup\limits_{\substack{u\in U,\xi\in\mathbb{R}^{p+1}\\ \|\xi\|=1}} \left\{\xi^T\mathcal{H}^u(t,x,\partial_t w, Dw, D^2w)\xi\right\} = 0 & t\in[0,T), x\in\mathbb{R}^d, z>0 \\ w(t,x,0) = w_0(t,x) & t\in[0,T), x\in\mathbb{R}^d \\ w(T,x,z) = \psi(x) - z & x\in\mathbb{R}^d, z\geq 0 \end{cases}$$

for a suitable definition of a symmetric matrix $\mathcal{H}^u$.

1.2.2 Asymptotic controllability under state constraints

The 4th chapter of the thesis concerns the study of the controllability properties of a stochastic system. In the present case, the notion of controllability is strictly connected with the stability properties. A dynamical system governed by an ordinary differential equation is said *stable* at a certain equilibrium point x_E if all the trajectories starting in a sufficiently small neighborhood of x_E remain forever close to x_E. A system is (locally) *asymptotically stable* if it is stable and, moreover, all the solutions starting near x_E converge to x_E. In the stochastic setting we will talk about stability in probability if the system satisfies the stability requirement with a certain probability. If the stochastic trajectories starting in a suitable neighborhood of the equilibrium x_E are stable and converge to x_E with probability one, the system is said almost surely asymptotically stable. In the controlled case the existence of a control guaranteeing the stability property is characterized by the notion of *controllability*. In the study of stability for nonlinear systems, a fundamental contribution is represented by the work of Alexander Lyapunov at the end of the 19th century. In his original work [137] of 1892, Lyapunov proposed two methods for proving the stability of deterministic nonlinear systems. In particular, the second method he presented relates the stability of the system with the existence of a suitable function, nowadays called a "Lyapunov function", satisfying a certain set of assumptions. In control theory this approach has been generalized thanks to the introduction of "Control-Lyapunov functions". Let be $x_E = 0$ an equilibrium point for a controlled system in $\mathbb{R}^d$

$$\dot{x} = b(x,u).$$

A Control-Lyapunov function is a continuously differentiable function $V : \mathbb{R}^d \to \mathbb{R}$ such that:

- V(0)=0;
- there exists $\delta > 0$ such that for any $x \neq 0$ with $|x| \leq \delta$ one has $V(x) > 0$ and $DV(x)\cdot b(x,u) \leq 0$ for some control u.

For the definition of non smooth control Lyapunov functions we may refer to [164, 165, 82, 155]. The second method of Lyapunov has been widely studied starting from the 60's [173, 105, 140, 123, 106, 125, 126] and it still represents one of the most used tools for studying stability.

Finding constructive procedures for defining Lyapunov functions represents an important domain of research in this field. An important result in this direction was obtained by Zubov in [174]. In this work the author proved that, for a suitable definition of a scalar function f, a Lyapunov function for the dynamical system defined above is given by the solution V of a particular kind of partial differential equation, called the Zubov equation:

$$DV(x) \cdot b(x) = -f(x)(1 - V(x))\sqrt{1 + \|b(x)\|^2}.$$

In particular Zubov proved that the domain of attraction of an asymptotically stable equilibrium x_E, defined by the states that converge to x_E, coincides with the set of points x such that $V(x) < 1$.
In the later years this method has been successfully applied in many contexts [1, 103, 118] and it has been generalized for dealing with different kind of systems, including control and stochastic systems [22, 105, 70, 72, 73, 69, 71, 67]. In particular the latter references establish a link between the Zubov method and optimal control theory. This link is based on the observation that Zubov's equation is a particular kind of HJB equation and it is thus naturally associated with an optimal control problem. The fact of looking at the Zubov method in the HJ framework has also the advantage of allowing the use of the available results in viscosity theory for dealing with the cases where the existence of smooth solutions cannot be guaranteed.

In particular, in [67], given a system of controlled stochastic differential equations, the Zubov method is used to characterize the following set

$$x \in \mathbb{R}^d : \quad \begin{array}{l} \text{for some choice of the control } u \\ X_x^u(t) \text{ converges to the equilibrium} \\ \text{with positive probability.} \end{array}$$

This set is called the domain of null-controllability for an asymptotically stable equilibrium point (or, more generally, for a set $\mathcal{T}$ replacing the equilibrium). It is proved in that paper that, for a suitable choice of a scalar function f, such domain corresponds to the set of points where the unique bounded viscosity solution of the second order problem

$$\begin{cases} \sup\limits_{u \in U} \left\{ -f(x,u)(1-V) - DV \cdot b(x,u) - \frac{1}{2} Tr[\sigma\sigma^T(x,u)D^2V] \right\} = 0 & x \in \mathbb{R}^d \setminus \mathcal{T} \\ V(x) = 0 & x \in \mathcal{T} \end{cases}$$

is strictly lower then one. The contribution given by this thesis is to introduce in this study the presence of state constraints, asking not only that the trajectories converge to $\mathcal{T}$ but also that they respect a prescribed constraint on the state. In other words we will be interested in the characterization and computation of the set of points

$$x \in \mathbb{R}^d : \quad \begin{array}{l} \text{for some choice of the control } u \\ X_x^u(t) \text{ converges to the equilibrium} \\ \text{and } X_x^u(t) \in \mathcal{K}, \forall t \geq 0 \\ \text{with positive probability.} \end{array}$$

In the deterministic case, the presence of state constraints has been considered in [104]. Partially inspired by this work, we will consider a stochastic optimal control problem of the form

$$v(x) = \inf_{u \in \mathcal{U}} \left\{ 1 + \mathbb{E}\left[\sup_{t \geq 0} \left(-e^{-\int_0^t f(X_x^u(s),u(s))ds - h(X_x^u(t))} \right) \right] \right\}$$

and we will prove in Theorem 4.4.1 that for suitable choices of the functions f and h the domain of asymptotic controllability coincides with the set of points such that $v(x) < 1$.

As already pointed out by the study developed in Chapter 3, dealing with maximum costs in the stochastic setting implies the necessity of introducing an auxiliary scalar variable y and leads to boundary problems with oblique derivatives conditions. We will prove in Theorem 4.5.4 that the value function ϑ associated with the auxiliary optimal control in dimension $d+1$, is a viscosity solution of the following Zubov-type HJB equation with mixed boundary conditions:

$$\begin{cases} \sup_{u\in U}\Big\{ -f(x,u)(1-\vartheta) - D\vartheta\cdot\tilde{b}(x,y,u) & x\in\mathbb{R}^d, -e^{-h(x)}<y<0 \\ \qquad -\frac{1}{2}Tr[\sigma\sigma^T(x,u)D_x^2\vartheta]\Big\} = 0 & \\ \vartheta(x,0)=1 & x\in\mathbb{R}^d \\ -\partial_y\vartheta = 0 & x\in\mathbb{R}^d, y=-e^{-h(x)}. \end{cases}$$

Despite of the analogies with the problem studied in Chapter 3, the particular feature of Zubov-type equations requires the use of particular tools. Indeed, because of the degeneracy of f near $\mathcal{T}$, new technical difficulties typical of this kind of equations arise for proving uniqueness results. In particular such a result cannot be proved by standard comparison arguments and the uniqueness statement of Corollary 4.6.5 will be obtained applying sub- and super-optimality principles.

1.2.3 Ergodic control and state constraints

In the last part of the thesis we present some preliminary results concerning the study of ergodic stochastic control problems in presence of state constraints.

Ergodic theory is devoted to the study of the long time behavior of dynamical systems. The type of information that is typically looked for is some result that relates the time average of a function along the trajectories only with its space average. More precisely, a deterministic dynamical system

$$\begin{cases} \dot{x} = b(x) \\ x(0) = x_0 \end{cases}$$

is said ergodic if there exists an *invariant measure* μ (that is a measure such that $\mu(\Phi_t(A)) = \mu(A)$, where Φ_t is the flux of the dynamical system) such that for any locally integrable function f

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(x(t))dt = \int f(z)d\mu(z)$$

for any initial data x_0. We can extend the previous property to the stochastic case requiring that

$$\lim_{T\to+\infty}\frac{1}{T}\mathbb{E}\Big[\int_0^T f(X(t))dt\Big] = \int f(z)d\mu(z).$$

We say that a stochastic process $X_x(\cdot)$ with law $p(y,t) := \mathbb{P}(X_x(t)\in dy)$ is ergodic in law if

$$\lim_{t\to+\infty} p(\cdot,t) \ = \bar{p}(\cdot)$$

for some function $\bar{p}$ independent on the initial distribution $p_0 = p(\cdot,0)$. Roughly speaking, this kind of properties say that no matter how the initial distribution is, after some time the system stabilizes "forgetting" its initial state.

In the case of optimal control problems over a long time period, the fact of minimizing a long-run average cost can be more suitable to answer to the desire of minimizing the performance on a long term and average basis. The study of ergodic control problems dates back to the late 50's. For what concerns the continuous case, among the first contributions we find [130, 169, 157]. Considered a certain criterion of performance J in the time interval $[0, T]$, studying the ergodic control problem means to study the minimization of the long-run average cost

$$\lim_{T \to +\infty} \frac{1}{T} J(T, x, u).$$

This kind of problems has been widely investigated in [6, 53, 54, 77, 76, 75, 7, 8, 156, 157, 151]. For our study we will make use of purely PDE techniques, analyzing the ergodicity of the problem by the ergodic properties of HJB equations. Denoted by $v(T, \cdot)$ the optimal value obtained minimizing over the set of controls $\mathcal{U}$ the quantity $J(T, \cdot, u)$, it is in fact well-known that $v(T, \cdot)$ can be characterized as the unique viscosity solution of a second order (first order in the deterministic case) HJB equation of the form

$$\partial_t v + H(x, Dv, D^2 v) = 0, \qquad t \in (0, T].$$

In order to be able to pass to the limit for T going to $+\infty$ in this equation, the main result we aim to prove is the uniform convergence of $\frac{1}{T} v(T, \cdot)$ and $v(T, \cdot) - v(T, 0)$ to a constant and to a function respectively. By using classical Abelian-Tauberian theorems, it is possible to prove that the convergence of $\frac{1}{T} J(T, \cdot, u)$ is equivalent to the convergence of the quantity $\lambda J_\lambda(\cdot, u)$ for λ going to 0, where $J_\lambda(\cdot, u)$ is the cost functional for an infinite horizon problem with discount factor $\lambda > 0$. Thanks to this observation in our study we directly considered this second case.

The main contribution given in Chapter 6 of the present thesis is to identify a particular class of ergodic problems that can be solved also taking into account the presence of state constraints. The state constrained ergodic problems studied in literature can be divided in three groups:

- the dynamics is periodic, that is the trajectories are constrained on a torus;
- the trajectories are reflected on the boundary of the state constraints;
- the presence of state constraints really act on the problem restricting the set of the admissible trajectories.

An overview of these three cases is presented in [77] for deterministic control systems. In the stochastic setting a study of the periodic case is given in [9, 4], whereas for the reflected dynamics, leading to HJB equations with Neumann boundary conditions, we may refer to [66, 52, 28, 43]. The third case with singular boundary conditions is considered in [131]. When the value function is the viscosity solution of the state constrained HJB equation

$$\begin{cases} \lambda v + H(x, Dv, D^2 v) = 0 & \text{in } int(\mathcal{K}) \\ \lambda v + H(x, Dv, D^2 v) \geq 0 & \text{in } \partial\mathcal{K}. \end{cases}$$

no results seem to be available at the moment. For this reason, we started studying the problem in a simplified setting assuming the invariance of the set of state constraints $\mathcal{K}$ with respect to our dynamics. This is formalized by the following hypothesis:

$$X_x^u(t) \in \mathcal{K}, \quad \forall t \geq 0 \quad \text{a.s.} \qquad \text{for any control } u \in \mathcal{U}.$$

Differently from what usually happens dealing with state constraints, for which only the super-solution property holds on the boundary, this assumption allows to prove that the value function v_λ solves an HJB equation in the whole domain $\mathcal{K}$. In this way, the analysis of the optimal control problem for λ fixed is much more simple. However even in this simplified framework the ergodic problem cannot be solved by the direct application of the techniques available in literature.
For this reason we introduce some further assumptions that provide at least a first suitable setting for solving the problem. In Theorem 6.4.2 the uniform convergence of λv_λ and $v_\lambda - v_\lambda(0)$ is proved under the following asymptotic flatness requirement:

$$\text{there exist } C_1 \geq 0, C_2 > 0 : \text{for any } u \in \mathcal{U}, x, y \in \mathcal{K}, t \geq 0$$
$$\mathbb{E}\left[|X_x^u(t) - X_y^u(t)|\right] \leq C_1 e^{-C_2 t}|x-y|.$$

This condition, appearing for the first time in [38] for studying ergodic control problems in the whole space, it is used to control the long time behavior of the stochastic trajectories. Denoted respectively by Λ and χ the constant and the function uniform limit of λv_λ and $v_\lambda(\cdot) - v_\lambda(0)$, by classical stability results for HJB equations we show that they solve the so-called *cell problem*

$$\Lambda + H(x, D\chi, D^2\chi) = 0 \qquad \text{in } \mathcal{K}.$$

Adding to this framework the following finiteness property of the trajectories

$$\sup_{t\geq 0} \sup_{x\in\mathcal{C}} \sup_{u\in\mathcal{U}} \mathbb{E}\left[|X_x^u(t)|\right] < \infty, \qquad \text{for any compact set } \mathcal{C} \subseteq \mathcal{K}$$

we finally prove in Theorem 6.4.3 that Λ is actually the unique constant that guarantees the existence of a viscosity solution for the limit HJB in a suitable class of functions.

1.2.4 Conclusions and perspectives

In this thesis we studied different stochastic control problems, proposing new solutions for facing the presence of state constraints.

Exploiting the link between optimal control and backward reachability we were able to deal with state constrained optimal control problems without making any controllability assumption on the dynamics. This was possible extending to the stochastic framework the level set method that has as the main advantage that one of allowing a particularly convenient treatment of the state constraints.
We analyzed the controllability (in probability) properties of stochastic diffusion systems. Generalizing the method of Zubov, we obtained a PDE characterization of the domain of asymptotic controllability with positive probability for a given system taking also into account the possible presence of state constraints.
In the last part of the thesis we dealt with state constrained stochastic ergodic problems finding out a suitable framework for proving the ergodicity of the problem.

This study leaded us to consider from the theoretical point of view different problems. We studied optimal control with maximum cost, for which we analyzed the numerical aspects and provided new error estimates. We dealt with the presence of unbounded controls obtaining a compactified characterization in terms of a generalized HJB equation.

On one hand this research opens the way to further developments in the field of applications since our main results, stated for general settings, can be adapted and tested on a wide class of problems representing a possible alternative to the commonly used techniques.
On the other hand the investigations carried out for the theoretical developments of our tools may contribute to enrich the existing literature on different subjects.

Chapter 2

Background

In this chapter we recall some classical results on stochastic optimal control problems and the associated Hamilton-Jacobi approach.

By the use of the Bellman's dynamic programming techniques the study of optimal control problems can be linked with the solution of a particular class of nonlinear partial differential equations: the Hamilton-Jacobi-Bellman equations. In particular first order equations are obtained dealing with deterministic control systems and second order equations appear in the stochastic case. In many situations the existence of classical solutions is not guaranteed. The suitable context for developing a complete existence and uniqueness theory turned out to be the viscosity solutions framework, introduced by Crandall and Lions in the early 80s.

The chapter is organized as follows. We start introducing a general formulation for stochastic optimal control problems with finite or infinite time horizon. The questions concerning the existence of optimal controls are addressed in Section 2.2. In Section 2.3 the Bellman's Dynamic Programming Principle is stated and its differential version, the Hamilton-Jacobi-Bellman equation, provided. The main definitions and tools in viscosity theory are given in Section 2.4 for the more general class of Hamilton-Jacobi equations. The invariance and viability aspects related to the presence of state constraints are presented in Section 2.5.

2.1 Stochastic optimal control problems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space, with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the canonical assumptions ($\mathcal{F}_0$ contains all the negligible sets, it is right-continuous, i.e $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ and it is left-continuous, i.e $\mathcal{F}_t = \mathcal{F}_{t-} := \sigma(\bigcup_{s<t} \mathcal{F}_s)$) and let $\mathcal{B}(\cdot)$ be a p-dimensional Brownian motion adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. We consider the following dynamics given by a system of controlled stochastic differential equations (SDEs) in $\mathbb{R}^d$:

$$(2.1.1) \qquad \begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))d\mathcal{B}(s) & s \in (t, T) \\ X(t) = x. \end{cases}$$

The elements that define a control system are: the horizon $T \in [0, +\infty]$, the initial time $t \in [0, T]$, the initial condition $x \in \mathbb{R}^d$, the *drift* $b : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$, the *volatility* $\sigma : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times p}$ and the control processes $u \in \mathcal{U}$, the set of progressively measurable processes taking almost surely (a.s.) values in $U \subseteq \mathbb{R}^m$.

We consider the following assumptions on U, b and σ:

(H_U) $$U \subset \mathbb{R}^m \text{ is a compact set ;}$$

(H_b) $$\begin{cases} (i) & b(\cdot,\cdot,\cdot) \text{ is continuous on } [0,T]\times\mathbb{R}^d\times U; \\ (ii) & \exists L_b \geq 0 \text{ such that } \forall x,y\in\mathbb{R}^d, t\in[0,T], u\in U: \\ & \quad |b(t,x,u)-b(t,y,u)| \leq L_b|x-y|; \end{cases}$$

(H_σ) $$\begin{cases} (i) & \sigma(\cdot,\cdot,\cdot) \text{ is continuous on } [0,T]\times\mathbb{R}^d\times U; \\ (ii) & \exists L_\sigma \geq 0 \text{ such that } \forall x,y\in\mathbb{R}^d, t\in[0,T], u\in U: \\ & \quad |\sigma(t,x,u)-\sigma(t,y,u)| \leq L_\sigma|x-y|. \end{cases}$$

It is well known (see for instance [170, Theorem 3.1]) that under these assumptions for any $(t,x)\in[0,T]\times\mathbb{R}^d$, $u\in\mathcal{U}$ there exists a unique solution to (2.1.1). We will denote this solution by $X^u_{t,x}(\cdot)$. It satisfies almost surely

$$X^u_{t,x}(\cdot) = x + \int_t^{\cdot} b(s, X^u_{t,x}(s), u(s))ds + \int_t^{\cdot} \sigma(s, X^u_{t,x}(s), u(s))d\mathcal{B}(s).$$

In the case of autonomous systems we will fix $t=0$ and the solution of (2.1.1) will be denoted by $X^u_x(\cdot)$.

We recall the following classical result in SDEs theory (see again [170, Theorem 2.4], for instance):

Proposition 2.1.1. *Under assumptions* (H_U),(H_b) *and* (H_σ)*, there exists a unique* $\{\mathcal{F}_t\}$*-adapted process* $X^u_{t,x}(\cdot)$ *strong solution of* (2.1.1)*. Moreover, let* $T<+\infty$*, there exists a constant* $C>0$ *(depending on* T, L_b *and* L_σ*), such that for any* $u\in\mathcal{U}$*,* $0\leq t\leq t'\leq T$ *and* $x,x'\in\mathbb{R}^d$

(2.1.2a) $$\mathbb{E}\left[\sup_{\theta\in[t,t']}\left|X^u_{t,x}(\theta)-X^u_{t,x'}(\theta)\right|^2\right] \leq C|x-x'|^2,$$

(2.1.2b) $$\mathbb{E}\left[\sup_{\theta\in[t,t']}\left|X^u_{t,x}(\theta)-X^u_{t',x}(\theta)\right|^2\right] \leq C(1+|x|^2)\ |t-t'|,$$

(2.1.2c) $$\mathbb{E}\left[\sup_{\theta\in[t,t']}\left|X^u_{t,x}(\theta)-x\right|^2\right] \leq C(1+|x|^2)\ |t-t'|.$$

A general optimal control problem is characterized by a *distributed cost* $\ell:[0,T]\times\mathbb{R}^d\times U\to\mathbb{R}$, a *terminal cost* $\psi:\mathbb{R}^d\to\mathbb{R}$ and a *discount factor* $\lambda\geq 0$ and it is defined by

(2.1.3) $$\inf_{u\in\mathcal{U}} \mathbb{E}\left[\int_t^T e^{-\lambda s}\ell(s, X^u_{t,x}(s), u(s))ds + e^{-\lambda T}\psi(X^u_{t,x}(T))\right].$$

We will consider the following assumptions:

(H_ℓ) $$\begin{cases} (i) & \ell(\cdot,\cdot,\cdot) \text{ is continuous on } [0,T]\times\mathbb{R}^d\times U; \\ (ii) & \exists L_\ell \geq 0 \text{ such that } \forall x,y\in\mathbb{R}^d, t\in[0,T], u\in U: \\ & \quad |\ell(t,x,u)-\ell(t,y,u)| \leq L_\ell|x-y|. \end{cases}$$

(H_ψ) ψ is a Lipschitz continuous function.

The value function is the map, denoted by v in this chapter, that associates to any $t \in [0,T]$ and $x \in \mathbb{R}^d$ the optimal value in (2.1.3). If $T < +\infty$ we will consider $\lambda = 0$ and (2.1.3) is called a *finite horizon optimal control problem*. We will say that the problem is in a "Mayer form" if $\ell \equiv 0$ and, otherwise, that it is a "Bolza problem". If $T = +\infty$, then we take $\lambda > 0$ and $\psi \equiv 0$ and we deal with an *infinite horizon optimal control problem*. For infinite horizon problems we will only consider the autonomous case, so the value function will depend only on the initial state x. On the basis of what we said one has

- **Finite horizon problem** $(T < \infty,\ \lambda = 0)$ **:**

$$v(t,x) = \inf_{u \in \mathcal{U}} \mathbb{E}\left[\int_t^T \ell(s, X^u_{t,x}(s), u(s))ds + \psi(X^u_{t,x}(T))\right] \quad \text{(Bolza)} \tag{2.1.4}$$

$$v(t,x) = \inf_{u \in \mathcal{U}} \mathbb{E}\left[\psi(X^u_{t,x}(T))\right] \quad \text{(Mayer)} \tag{2.1.5}$$

- **Infinite horizon problem** $(\psi \equiv 0,\ \lambda > 0)$ **:**

$$v(x) = \inf_{u \in \mathcal{U}} \mathbb{E}\left[\int_t^{+\infty} e^{-\lambda s}\ell(X^u_{t,x}(s), u(s))ds\right].$$

Remark 2.1.2. It is easy to show that, by adding a state variable following the dynamics given by the running cost, any finite horizon problem can be transformed in a Mayer's type problem ([170, Remark 3.2.(*iii*)]).

2.2 On the existence of optimal controls

The definition of optimal control problems given in the previous section is referred in literature with the name of "strong formulation". In this case the probability space and the filtration are fixed at the beginning and a control is a progressively measurable process with respect to this filtration.
On the other hand it is also possible to give a "weak formulation" of the general optimal control problem (2.1.3) allowing to change the probability space. If such a weak formulation is considered, a control is given by a 6-tuple

$$\pi \equiv (\Omega_\pi, \mathcal{F}_\pi, \{F_{\pi,t}\}_{t\geq 0}, \mathbb{P}_\pi, \mathcal{B}_\pi(\cdot), u_\pi(\cdot))$$

such that

- $(\Omega_\pi, \mathcal{F}_\pi, \{F_{t,\pi}\}_{t\geq 0}, \mathbb{P}_\pi)$ is a filtered probability space ;
- $\mathcal{B}_\pi(\cdot)$ is a p-dimensional Brownian motion;
- $u_\pi(\cdot)$ is a progressively measurable process with respect to $\{\mathcal{F}_{\pi,t}\}_{t\geq 0}$ taking value in $U \subseteq \mathbb{R}^m$.

We will denote by Π the set of the 6-tuples π satisfying these assumptions. Under this formulation the optimal control problems we aim to solve is

$$\inf_{\pi \in \Pi} \mathbb{E}_\pi\left[\int_t^T e^{-\lambda s}\ell(s, X^u_{t,x}(s), u(s))ds + e^{-\lambda T}\psi(X^u_{t,x}(T))\right] \tag{2.2.1}$$

where $\mathbb{E}_\pi$ denotes the expectation with respect to the probability measure $\mathbb{P}_\pi$.

An important issue in optimal control theory is to prove the existence of optimal controls for problems (2.1.3) and (2.2.1). In the deterministic case ($\sigma(x,u) \equiv 0$) the existence of optimal controls can be proved by compactness arguments under some convexity assumptions that enable to apply the Filippov measurable selection theorem (see for instance [81, Theorem 23.2]). In the stochastic framework and under a strong formulation of the optimal control problem it is in general not possible to apply similar techniques because of the lack of a compact structure on the set of stochastic trajectories. A special case occurs when the drift b and the volatility σ are linear functions of the state and the control variable. In this case, the particular structure of the problem, allows to rigorously prove an existence result that we state below as presented in [172, Chapter II Theorem 5.2]. In the next chapters of the thesis some easy applications of this result will be discussed (see Chapter 3, Theorem 3.9.2 and Chapter 5, Theorem 5.9.2).
Let us consider the following stochastic linear controlled system

$$\begin{cases} dX(t) = (AX(t) + Bu(t))dt + (CX(t) + Du(t))d\mathcal{B}(t) & t \in (0,T) \\ X(0) = x. \end{cases} \tag{2.2.2}$$

where A, B, C and D are matrices of suitable sizes and $\mathcal{B}(\cdot)$ is a one-dimensional Brownian motion. The finite horizon Bolza problem (2.1.4) is considered (with $t = 0$ and ℓ independent of t).

Theorem 2.2.1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space and $X^u_x(\cdot)$ be the solution of (2.2.2) associated to $u(\cdot) \in L^2(0,T;U)$ (with $U \subseteq \mathbb{R}^m$). Let us assume that the functions ℓ and ψ and the set U of control values are convex. If problem (2.1.4) is finite and either U is compact or U is closed and $\ell(x,u) \geq C_1|u|^2 - C_2$, $\psi(x) \geq -C_2, \forall (x,u) \in \mathbb{R}^d \times U$ for some constants C_1 and C_2, then (2.1.4) admits an optimal control.*

In the general case of stochastic nonlinear control systems, it was pointed out by several authors that the most suitable framework for proving existence is the weak one (see [94, 107, 41, 127]). The result we report below take into account this more general setting under the assumption (H_U) of compactness of the set of control values. Generalizations of this result to the case of unbounded controls have been proposed in [107] and [135]. One has the following result ([172, Chapter II, Theorem 5.3]):

Theorem 2.2.2. *Let assumptions (H_U),(H_b),(H_σ),(H_ψ) and (H_ℓ) be satisfied. Let us also assume that for every $t \in [0,T], x \in \mathbb{R}^d$*

$$(b, \sigma\sigma^T, \ell)(t,x,U) := \Big\{(b_i(t,x,u), (\sigma\sigma^T)_{ij}(t,x,u), \ell(t,x,u)), \\ i = 1,\dots,d, j = 1,\dots,p, u \in U\Big\}$$

is a convex set. The weak formulation of problem (2.1.4) is taken into account. If such a problem is finite, then it admits an optimal control in Π.

2.3 Dynamic Programming approach for stochastic optimal control problems

The main idea of the dynamic programming approach is that the value function v satisfies a functional equation, that is called the *Dynamic Programming Principle* (DPP). It can be stated as follows

Theorem 2.3.1 (DPP). *Assume* (H_U),(H_b),(H_σ),(H_ℓ) *and* (H_ψ). *Then :*

(*i*) *Finite horizon Bolza problem: for any* $(t,x) \in [0,T) \times \mathbb{R}^d$ *and for any* $\{\mathcal{F}_t\}$*-stopping time* θ *with values in* $[t,T]$

$$v(t,x) = \inf_{u \in \mathcal{U}} \mathbb{E}\left[v(\theta, X^u_{t,x}(\theta)) + \int_t^\theta \ell(s, X^u_{t,x}(s), u(s))ds\right]$$

(*ii*) *Infinite horizon problem: for any* $x \in \mathbb{R}^d$ *and for any* $\{\mathcal{F}_t\}$*-stopping time* $\theta \geq 0$

$$v(x) = \inf_{u \in \mathcal{U}} \mathbb{E}\left[e^{-\lambda\theta} v(X^u_x(\theta)) + \int_0^\theta e^{-\lambda t}\ell(X^u_x(t), u(t))dt\right].$$

If v is two times differentiable, by using the Itô formula it is possible to prove that v is a solution of the following equation:

- **Finite horizon problem:**

$$(2.3.1) \qquad \begin{cases} -\partial_t v(t,x) + H(t,x,D_x v, D^2_x v) = 0 & t \in [0,T), x \in \mathbb{R}^d \\ v(T,x) = \psi(x) & x \in \mathbb{R}^d \end{cases}$$

- **Infinite horizon problem:**

$$(2.3.2) \qquad \lambda v(x) + H(x, Dv, D^2 v) = 0 \qquad x \in \mathbb{R}^d$$

where we denoted by Dv and $D^2 v$ respectively the gradient and the Hessian matrix of v. The function $H : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}^d \to \mathbb{R}$ (with $\mathcal{S}^d$ we denote the set of $d \times d$ symmetric matrices), namely the Hamiltonian of the system, is defined by

$$H(t,x,p,Q) := \sup_{u \in U}\left\{ -b(t,x,u)\cdot p - \frac{1}{2}Tr[\sigma\sigma^T(t,x,u)Q] - \ell(t,x,u)\right\}$$

for any $t \in [0,T], x, p \in \mathbb{R}^d$ and $Q \in \mathcal{S}^d$ (the definition for the autonomous case follows trivially). We point out that for deterministic control system, corresponding to the case $\sigma \equiv 0$, equations (2.3.1) and (2.3.2) reduce to first order equations. Because of the possible degeneracy of the volatility σ, the existence of smooth solutions to equation (2.3.1) (or (2.3.2)) cannot be guaranteed and for this reason solutions have to be considered in some weak sense.

2.4 Characterization via Viscosity Solutions Theory

The suitable framework for dealing with equations (2.3.2) and (2.3.1) is the *viscosity solutions theory.* This theory provides existence and uniqueness for a more general class of fully nonlinear equations, known with the name of Hamilton-Jacobi (HJ) equations, that can be written in following form

$$(2.4.1) \qquad F(x, v, Dv, D^2 v) = 0, \qquad x \in D$$

where D is an open set in $\mathbb{R}^d$.

Remark 2.4.1. It is clear that equations (2.3.1) and (2.3.2), that we refer with the name of Hamilton-Jacobi-Bellman (HJB) equations, can be included in the general formulation (2.4.1). Indeed equation (2.3.2) corresponds to the case $D = \mathbb{R}^d$ and

$$F(x, r, p, Q) := \lambda r + H(x, p, Q)$$

and the time dependent case of equation (2.3.1) to $D = (0, T) \times \mathbb{R}^d$ and

$$F((t, x), r, (p_1, p), Q) := -p_1 + H(t, x, p, (Q_{ij})_{i,j \geq 2}).$$

In what follows the abbreviation USC (resp. LSC) stands for upper semi-continuous (resp. lower semi-continuous).

Definition 2.4.2 (Viscosity solutions)**.** An USC (resp. LSC) function v on D is a *viscosity sub-solution* (resp. *super-solution*) of (2.4.1), if for each function $\varphi \in C^2(D)$, at each maximum (resp. minimum) point x of $v - \varphi$ the following inequalities hold

$$F(x, \varphi, D\varphi, D^2\varphi) \leq 0 \qquad \text{on} \quad D$$

(resp.

$$F(x, \varphi, D\varphi, D^2\varphi) \geq 0 \qquad \text{on} \quad D).$$

Finally a continuous function v is a viscosity solution of (2.4.1) if it is both a sub- and super-solution.

Viscosity solutions were introduced, in the nowadays formulation, by Crandall and Lions in 1983 in their famous paper [86] (see also [85] and [97, 96] for earlier contributions). Even if the main definitions given in this paper can be extended to the second order case, only first order equations are taken into account. In the same year Lions in [132] and [133] characterizes the value function associated to optimal control problems for controlled diffusion (as (2.1.3)) as the unique continuous viscosity solution of a second order HJB equation. However the uniqueness result presented in these papers is based on ad hoc arguments strictly connected with the particular form of the equation.

In the general case uniqueness is obtained as a consequence of some comparison result between sub- and super-solutions, that establishes that for any sub-solution $\underline{v}$ and any super-solution $\overline{v}$

$$\underline{v}(x) \leq \overline{v}(x) \qquad \text{on } D.$$

Therefore uniqueness follows from the fact that a solution v is at the same time a sub- and a super-solution. The classical method for proving comparison results for first order equations ($\sigma \equiv 0$), is based on the doubling variable technique (see [86] for instance). It consists in the definition of the following family of auxiliary functions

$$\Phi_\varepsilon(x, y) := \underline{v}(x) - \overline{v}(y) - \frac{|x - y|^2}{\varepsilon}, \qquad \varepsilon > 0.$$

Given some maximum point $(x_\varepsilon, y_\varepsilon)$ for Φ_ε the result is proved considering $x \to \overline{v}(y_\varepsilon) + \frac{|x-y_\varepsilon|^2}{\varepsilon}$ and $y \to \underline{v}(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{\varepsilon}$ as test functions for $\underline{v}(x)$ and $\overline{v}(y)$ respectively. However the same technique does not apply to the second order equations, since the information given by these test functions on the second order derivatives is not sufficient to get the result. For several years no uniqueness results were available for fully nonlinear second order equations in the general form (2.4.1). A first important achievement in this direction was obtained in 1988 by Jensen, who provided in [115] a comparison principle for a family of equations (2.4.1) independent of x using "supconvolution" and "infconvolution" approximations. In the later years a wider class of problems was taken into account including also boundary value problems. We mention, among the main contributions [110, 83, 113, 109, 111, 116]. In this framework it turned out to be particularly useful to introduce another equivalent definition of viscosity solution making use of the concept of *semijets*.

Definition 2.4.3 (Semijets)**.** Let v be an USC function. The *superjet* of v at some point x is

$$\begin{aligned}\mathcal{J}^{2,+}v(x) :=& \big\{(p,X) \in \mathbb{R}^d \times \mathcal{S}^d : \\ & v(y) \le v(x) + p\cdot(y-x) + \frac{1}{2}X(y-x)\cdot(y-x) + o\|x-y\|^2\big\}.\end{aligned}$$

We also define its closure

$$\begin{aligned}\overline{\mathcal{J}}^{2,+}v(x) :=& \big\{(p,X) \in \mathbb{R}^d \times \mathcal{S}^d : \\ & \exists x_n \to x, \exists (p_n, X_n) \in \mathcal{J}^{2,+}v(x_n) \text{ s.t. } (p_n, X_n) \to (p,X)\big\}.\end{aligned}$$

Analogously for a LSC function v we define the *subjet*

$$\begin{aligned}\mathcal{J}^{2,-}v(x) :=& \big\{(p,X) \in \mathbb{R}^d \times \mathcal{S}^d : \\ & v(y) \ge v(x) + p\cdot(y-x) + \frac{1}{2}X(y-x)\cdot(y-x) + o\|x-y\|^2\big\}\end{aligned}$$

and its closure

$$\begin{aligned}\overline{\mathcal{J}}^{2,-}v(x) :=& \big\{(p,X) \in \mathbb{R}^d \times \mathcal{S}^d : \\ & \exists x_n \to x, \exists (p_n, X_n) \in \mathcal{J}^{2,-}v(x_n) \text{ s.t. } (p_n, X_n) \to (p,X)\big\}.\end{aligned}$$

It is easy to observe that $\mathcal{J}^{2,+}v(x) = -\mathcal{J}^{2,-}(-v(x))$. Essentially these objects extend to the second order case the notion, largely used in nonsmooth analysis, of sub and super differential leading to a definition of Dv and D^2v in the nonsmooth case. We can consider the following definition of viscosity solution

Definition 2.4.4. An USC (resp. LSC) function v on D is a *viscosity sub-solution* (resp. *super-solution*) of (2.4.1) if

$$F(x,v,p,X) \le 0 \qquad \text{for any } x \in D, (p,X) \in \overline{\mathcal{J}}^{2,+}v(x)$$

(resp.

$$F(x,v,p,X) \ge 0 \qquad \text{for any } x \in D, (p,X) \in \overline{\mathcal{J}}^{2,-}v(x)).$$

Finally a continuous function v is a viscosity solution of (2.4.1) if it is both a sub- and super-solution.

Equivalence between Definitions 2.4.2 and 2.4.4 can be shown (see, for instance, [172, Proposition 5.6]). We also report here a key result that is often very useful for proving comparison results: the so-called "Crandall-Ishii lemma". It avoids the explicit regularization by convolution of the solutions. We give the statement of the result in its most general formulation as given in [84, Theorem 3.2]:

Lemma 2.4.5 (Crandall-Ishii lemma). *Let D_i be a locally compact subset of $\mathbb{R}^{d_i}$ for $i = 1, \ldots, k$, $D := D_1 \times \ldots \times D_k \subseteq \mathbb{R}^d$ $(d = d_1 + \ldots + d_k)$, $v_i \in USC(D_i)$ and φ be a twice continuously differentiable function in a neighborhood of D. Set*

$$v(x) := v_1(x_1) + \ldots + v_k(x_k)$$

for $x \equiv (x_1, \ldots, x_k) \in D$ and suppose that $\hat{x} \in D$ is a local maximum point for $v - \varphi$. Then for any $\alpha > 0$ there exists $X_i \in \mathcal{S}^{d_i}$ such that

$$(D_{x_i}\varphi(\hat{x}), X_i) \in \overline{\mathcal{J}}^{2,+} v_i(\hat{x}_i), \qquad \text{for } i = 1, \ldots, k$$

and the following matrix inequalities hold

$$-\left(\frac{1}{\alpha} + \|D^2\varphi(\hat{x})\|\right) I_d \leq \begin{pmatrix} X_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & X_k \end{pmatrix} \leq D^2\varphi(\hat{x}) + \alpha(D^2\varphi(\hat{x}))^2$$

(where for a matrix $A \in \mathcal{S}^d$ we define $\|A\| := \sup\{|A\xi \cdot \xi| : |\xi| \leq 1\}$).

For the special case of parabolic HJ equations will be also useful to introduce the parabolic semijets:

Definition 2.4.6 (Parabolic semijets). Let $v : [0,T] \times D \to \mathbb{R}$ an USC function. The *parabolic superjet* of v at some point (t,x) is

$$\begin{aligned}\mathcal{P}^{1,2,+} v(t,x) :=\big\{&(a,p,X) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d : \\ &v(s,y) \leq v(t,x) + a(s-t) + p \cdot (y-x) + \frac{1}{2} X(y-x) \cdot (y-x) \\ &+ o(|s-t| + \|x-y\|^2)\big\}.\end{aligned}$$

Let $v : [0,T] \times D \to \mathbb{R}$ a LSC function. The *parabolic subjet* of v at some point (t,x) is

$$\begin{aligned}\mathcal{P}^{1,2,-} v(t,x) :=\big\{&(a,p,X) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d : \\ &v(s,y) \geq v(t,x) + a(s-t) + p \cdot (y-x) + \frac{1}{2} X(y-x) \cdot (y-x) \\ &+ o(|s-t| + \|x-y\|^2)\big\}.\end{aligned}$$

The closures $\overline{\mathcal{P}}^{1,2,+} v(t,x), \overline{\mathcal{P}}^{1,2,-} v(t,x)$ of these sets are defined as done in the elliptic case.
We also state below the parabolic version of the Crandall-Ishii lemma [84, Theorem 8.3]:

Lemma 2.4.7 (Crandall-Ishii lemma, parabolic version). *Let D_i be a locally compact subset of $\mathbb{R}^{d_i}$ for $i = 1, \ldots, k$, $D := D_1 \times \ldots \times D_k \subseteq \mathbb{R}^d$ $(d = d_1 + \ldots + d_k)$ and $v_i \in USC((0,T) \times D_i)$. Let φ be defined on an open neighborhood of $(0,T) \times D$, once continuously differentiable in t and twice continuously differentiable in $x = (x_1, \ldots, x_k)$. Set*

$$v(t,x) := v_1(t,x_1) + \ldots + v_k(t,x_k)$$

and suppose that $(\hat{t}, \hat{x}) \in (0,T) \times D$ is a local maximum point for $v - \varphi$. Assume moreover that there is an $r > 0$ such that for every $M > 0$ there is a C such that for every $i = 1, \cdots, k$

$$\begin{aligned}&a_i \leq C \text{ whenever } (a_i, p_i, X_i) \in \overline{\mathcal{P}}^{1,2,+} v_i(\hat{t}, \hat{x}_i), \\ &|x_i - \hat{x}_i| + |t - \hat{t}| \leq r \text{ and } |v_i(t,x_i)| + |p_i| + \|X_i\| \leq M.\end{aligned}$$

Then for any $\alpha > 0$ there exist $a_i \in \mathbb{R}$, $X_i \in \mathcal{S}^{d_i}$ such that

$$(a_i, D_{x_i}\varphi(\hat{t},\hat{x}), X_i) \in \overline{\mathcal{P}}^{1,2,+} v_i(\hat{t},\hat{x}_i), \qquad \textit{for } i = 1,\dots,k$$

$$a_1 + \ldots + a_k = \partial_t \varphi(\hat{t},\hat{x})$$

and the following matrix inequalities hold

$$-\left(\frac{1}{\alpha} + \|D_x^2\varphi(\hat{t},\hat{x})\|\right) I_d \leq \begin{pmatrix} X_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & X_k \end{pmatrix} \leq D_x^2\varphi(\hat{t},\hat{x}) + \alpha(D_x^2\varphi(\hat{t},\hat{x}))^2.$$

We refer to [84] and the many references therein for a very useful survey containing a wide choice of comparison results.

2.5 Viable and invariant sets of state constraints

As mentioned in the introduction, the aim of this thesis is to address different issues that arise dealing with stochastic control systems in presence of state constraints. In the stochastic setting the concept of state constraint can be considered in different ways (probability constraints, almost sure constraints, constraints in expectation). In this section we only take into account the strongest request, that is that the constraints are satisfied almost surely. Considering state constrained problems, two fundamental concepts are those of viability and invariance.

Let $\mathcal{K}$ be a nonempty set in $\mathbb{R}^d$. We consider here the autonomous version of the control system (2.1.1)

$$\text{(2.5.1)} \qquad \begin{cases} dX(t) = b(X(t), u(t))ds + \sigma(X(t), u(t))d\mathcal{B}(t) \quad t > 0 \\ X(0) = x \end{cases}$$

and we denote by $X_x^u(\cdot)$ the trajectory starting at point x and associated to the control $u \in \mathcal{U}$. Roughly speaking we will say that $\mathcal{K}$ is viable if for all the initial states $x \in \mathcal{K}$ there exists at least one trajectory that remains in $\mathcal{K}$ for all $t \geq 0$ almost surely. The set $\mathcal{K}$ is invariant if all the trajectories starting in $\mathcal{K}$ remain in $\mathcal{K}$ forever in $\mathcal{K}$ almost surely. Clearly every invariant set is also viable. One has:

Definition 2.5.1. A nonempty set $\mathcal{K}$ is said to be *viable* for the controlled stochastic differential equation (2.5.1) if

$$\forall x \in \mathcal{K}, \text{there exists } u \in \mathcal{U} : X_x^u(t) \in \mathcal{K},\ \forall t \geq 0 \text{ a.s. } .$$

$\mathcal{K}$ is said to be *invariant* if for any $x \in \mathcal{K}$ one has

$$X_x^u(t) \in \mathcal{K},\ \forall t \geq 0 \text{ a.s. } , \forall u \in \mathcal{U}.$$

Dealing with state constrained optimal control problems the aim is to minimize (or maximize) some cost functional over the set of control strategies $\mathcal{U}$ asking at the same time that some state constraints is satisfied. This naturally restricts the set of admissible controls to the set of the controls $u \in \mathcal{U}$ such that the constraints are satisfied. In particular for the case of state constraints that are required to be satisfied almost surely, the set of admissible controls for a certain point $x \in \mathcal{K}$ becomes

$$\mathcal{U}_{\mathcal{K}}(x) := \Big\{ u \in \mathcal{U} : X_x^u(t) \in \mathcal{K}, \forall t \geq 0 \text{ a.s. } \Big\}.$$

It is clear that this set will be empty at some point $x \in \mathcal{K}$ if the set $\mathcal{K}$ is not viable. On the other hand we will have $\mathcal{U}_{\mathcal{K}}(x) = \mathcal{U}$ for any $x \in \mathcal{K}$ if $\mathcal{K}$ is invariant.

There is a wide literature concerning the characterization of viable and invariant sets. In the deterministic case, $\sigma \equiv 0$, the first reference is the work of Nagumo [146] for uncontrolled systems of ordinary differential equations. Classical references in the framework of control systems and differential inclusions are [17, 18]. Sufficient conditions for invariance of closed sets in the case of uncontrolled diffusions, $\sigma \not\equiv 0$, were firstly given in the book of Friedman [102] (observe that in the uncontrolled case the concept of viability and invariance coincide). In the later years this kind of results have been extended, by the use of different techniques, to characterize the invariance and viability properties of controlled diffusions (we may refer to [65, 25, 31, 13, 19] and [14]). We report below a characterization of invariant and viable closed sets in $\mathbb{R}^d$ based on the notion of second order normal cone.

Definition 2.5.2. Let $\mathcal{K} \subset \mathbb{R}^d$ a closed set. The *second order normal cone* at some point $x \in \mathcal{K}$ is

$$\mathcal{N}_{\mathcal{K}}^2(x) := \Big\{ (p, X) \in \mathbb{R}^d \times \mathcal{S}^d : \text{for } \mathcal{K} \ni y \to x \\ p \cdot (y - x) + \frac{1}{2}(y - x) \cdot X(y - x) \geq o(|y - x|^2) \Big\}.$$

Theorem 2.5.3. *Let assumptions* (H_b),(H_σ) *and* (H_U) *be satisfied and let* $\mathcal{K} \subseteq \mathbb{R}^d$ *be a closed set. The set* $\mathcal{K}$ *is invariant for the system* (2.5.1) *if and only if for any* $x \in \partial\mathcal{K}$ *and* $(p, X) \in \mathcal{N}_{\mathcal{K}}^2(x)$*, it holds:*

$$b(x, u) \cdot p + \frac{1}{2} Tr[\sigma\sigma^T(x, u)X] \geq 0, \qquad \forall u \in U. \tag{2.5.2}$$

Let us assume, in addition, that the set $\{(b, \sigma\sigma^T)(t, x, U)\}$ *is convex. Then the set* $\mathcal{K}$ *is viable if and only if for any* $x \in \partial\mathcal{K}$ *and* $(p, X) \in \mathcal{N}_{\mathcal{K}}^2(x)$

$$\exists u \in U : \qquad b(x, u) \cdot p + \frac{1}{2} Tr[\sigma\sigma^T(x, u)X] \geq 0. \tag{2.5.3}$$

We refer to [25] (for the invariance property) and to [26] (for the viability property).

We stress here that although questions concerning the viability properties of the sets naturally arise dealing with state constraints, it is not in the aims of this thesis to investigate such properties. With the exception of Chapter 6, where for technical reasons we will restrict the problem in an invariant domain, the tools we develop work in the same way independently of the viability properties satisfied by the set of state constraints. By the way this kind of properties can be easily derived by the outcome of our results.

Chapter 3

Reachability analysis under state-constraints

Related publications:

O. Bokanowski, A. Picarelli and H. Zidani, *Dynamic Programming and Error Estimates for Stochastic Control Problems with Maximum Cost*, Appl. Math. Optim., DOI: 10.1007/s00245-014-9255-3 (2014), pp. 1-39

O. Bokanowski, A. Picarelli, *Reachablity for state constrained stochastic control problems*, 3 pages extended abstract, Proceedings of the 20th MTNS conference, Melbourne, Australia, 9-13 July 2012.

3.1 Introduction

In this chapter we deal with the characterization and computation of the backward reachable set for a system of controlled diffusions in presence of state constraints. Let $\mathcal{T} \subseteq \mathbb{R}^d$ be a target set and let $\mathcal{K} \subseteq \mathbb{R}^d$ be a set of state constraints. Let $T \in [0, +\infty)$ be a fixed time horizon. As in the previous chapter, $X^u_{t,x}(\cdot)$ denotes the solution of a system of controlled stochastic differential equations associated with the control $u \in \mathcal{U}$, starting at time t from the position $x \in \mathbb{R}^d$.

We aim to characterize the set of all the initial points $x \in \mathbb{R}^d$ from which it is almost surely possible to reach the target $\mathcal{T}$ at the final instant T, satisfying the state constraints in the whole interval $[t, T]$. The set of such points will be called the *state constrained backward reachable set* and it will be denoted by $\mathcal{R}^{\mathcal{T},\mathcal{K}}_t$, i.e.

$$\mathcal{R}^{\mathcal{T},\mathcal{K}}_t := \Big\{ x \in \mathbb{R}^d : \exists u \in \mathcal{U} \text{ such that} \\ \Big(X^u_{t,x}(T) \in \mathcal{T} \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t, T] \Big) \text{ a.s.} \Big\}.$$

Because of their importance in applications ranging from engineering to biology and economics, questions of reachability have been studied extensively in the control literature (see [47, 121, 139, 138, 162, 16]). Stochastic target problems arising in finance have been also analyzed in [163, 61, 56] where, by establishing a geometric dynamic programming

principle, it is proved that a partial differential equation (the analogue of the Hamilton-Jacobi-Bellman equation for this problem) is satisfied by the reachable sets.
Here, we suggest to characterize the reachable sets by using a level set method (see [147] for its earlier introduction). At the basis of this approach there is the idea of looking at the set $\mathcal{R}_t^{\mathcal{T},\mathcal{K}}$ as a level set of a continuous function that can be characterized as the unique solution, in some sense, of a partial differential equation. Recently, the level set approach has been extended to deal with general deterministic optimal control problems in presence of state constraints, see for example [47, 2]. Applying this method we will see in Section 3.2 the interest of dealing with stochastic optimal control problems in the following "maximum form":

$$\inf_{u\in\mathcal{U}} \mathbb{E}\left[\psi\left(X_{t,x}^u(T),\ \max_{s\in[t,T]} g(X_{t,x}^u(s))\right)\right]. \tag{3.1.1}$$

This problems also arise from the study of some path-dependent options in finance (lookback options) and, motivated by this application, they have been studied in [29, 37, 30]. For this reason, starting from Section 3.4 the chapter is focused on the study of stochastic optimal control problems in the general form (3.1.1). Particular attention will be devoted to the numerical aspects including convergence analysis and error estimates.

The chapter is organized as follows: Section 3.2 is devoted to a general discussion on stochastic reachability problems, presenting a short overview of the main references available in literature. Section 3.3 explains how the level set approach can be applied to stochastic state constrained reachability problems. In particular it will be shown how it is possible to connect a state constrained reachability problem to an optimal control problem of the form (3.1.1). Stochastic optimal control problems with cost depending on a maximum will be presented in Section 3.4. Section 3.5 is devoted to the characterization of the value function associated with this kind of problems by the suitable HJB equation. In Section 3.6 the numerical approximation is discussed and a general convergence result is provided. The semi-Lagrangian scheme is presented in Section 3.6.2 and the properties of this scheme are investigated. In Section 3.7 error estimates for a semi-Lagrangian scheme are presented. Finally some numerical tests are presented in Section 3.8 to analyze the relevance of the proposed scheme.

3.2 On stochastic reachability

Before extensively discuss the application of the level set approach for solving reachability problems we briefly present in this section some of the main contributions to the study of this kind of problems available in literature.
An important part of literature is due to the works of Soner, Touzi, Bouchard et al. (see [163, 162, 56, 61]). At a first stage the authors were interested in the characterization of the value function ζ_Φ solution of the following particular optimal control problem

$$\zeta_\Phi(t,x) := \inf\left\{ z\in\mathbb{R} : \exists u\in\mathcal{U} \text{ such that } Z_{t,x,z}^u(T) \geq \Phi(X_{t,x}^u(T)) \text{ a.s.} \right\} \tag{3.2.1}$$

where the process $(X_{t,x}^u(\cdot), Z_{t,x,z}^u(\cdot))$ in $\mathbb{R}^d\times\mathbb{R}$ solves a SDE of the following form:

$$\begin{cases} dX(s) = b_X(s,X(s),u(s))ds + \sigma_X(s,X(s),u(s))d\mathcal{B}(s) \\ dZ(s) = b_Z(s,X(s),Z(s),u(s))ds + \sigma_Z(s,X(s),Z(s),u(s))d\mathcal{B}(s) \\ (X(t),Z(t)) = (x,z)\in\mathbb{R}^{d+1}. \end{cases} \tag{3.2.2}$$

It is clear that the characterization of ζ_Φ is closely related to the solution of a particular reachability problem (as defined in the previous section) where the target $\mathcal{T}$ is given by the epigraph of the function Φ. The study of this kind of problems was originally motivated by applications in finance. Indeed for some particular choice of the coefficients $b_X, \sigma_X, b_Z, \sigma_Z$, (3.2.1) is a super-replication problem (see [88, 95, 90, 89] for instance) where the value function ζ_Φ represents the minimal initial capital which allows, for a suitable choice of the strategy $u \in \mathcal{U}$, to hedge the contingent claim given by the payoff $\Phi(X^u_{t,x}(T))$. In [163] is obtained a characterization of the value function ζ_Φ as a discontinuous solution of a particular HJB equation by proving that ζ_Φ satisfies the following non classical geometric DPP:

$$\zeta_\Phi(t,x) = \inf\left\{ z \in \mathbb{R} : \exists u \in \mathcal{U} \text{ such that } Z^u_{t,x,z}(T) \geq \zeta_\Phi(\theta, X^u_{t,x}(\theta)) \right\}$$

for any $[t,T]$-valued stopping time θ (see also [159]). In [56] this kind of approach is generalized taking into account also mixed diffusion processes.
It is important to remark that stochastic target problems as (3.2.1) are also closely related to the theory of forward-backward SDEs (see for instance [5, 148, 149, 89]). We can in fact see (3.2.1) as the problem of finding $(X(\cdot), Z(\cdot), u)$ with the minimal $Z(0)$ such that $u \in \mathcal{U}$, $X(0) = x$ is fixed and $(X(\cdot), Z(\cdot))$ is solution of a forward-backward SDE (in particular X will be associated to the forward component and Z to the backward part). See for more details [89].

General stochastic target problems (i.e. target problems where $\mathcal{T}$ has not the form of an epigraph) are finally considered in [162]. In this case the authors modify the geometric techniques developed in the previous cases in order to deal with the characterization of the set

$$\mathcal{R}^{\mathcal{T}}_t := \left\{ x \in \mathbb{R}^d : \exists u \in \mathcal{U} \text{ such that } X^u_{t,x}(T) \in \mathcal{T} \text{ a.s.} \right\}$$

(it coincides exactly with the definition of the backward reachable set we gave in the introduction in absence of state constraints). For any stopping time θ taking values in $[t,T]$, the geometric DPP satisfied by $\mathcal{R}^{\mathcal{T}}_t$ takes the form

$$\mathcal{R}^{\mathcal{T}}_t = \left\{ x \in \mathbb{R}^d : \exists u \in \mathcal{U} \text{ such that } X^u_{t,x}(\theta) \in \mathcal{R}^{\mathcal{T}}_\theta \text{ a.s.} \right\}.$$

We conclude the section mentioning that target problems, with or without state constraints, can also be studied in the framework of the viability theory. The state constrained backward reachable set $\mathcal{R}^{\mathcal{T},\mathcal{K}}_0$ coincides in fact with what, in the context of deterministic viability, is called the *T-exact capture basin* (see [21, Definition 4.3.1]). We refer to [21] and the references therein for the main results in the deterministic framework. In the case of diffusion processes we may refer to [14] and [20].

3.3 The level set approach for state constrained stochastic reachability problems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a càdlàg process Z with values in $\mathbb{R}^d$ ($d \geq 1$) and independent increments. Given the time horizon $T > 0$, we denote by $\mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}$ the completion of the natural filtration induced by Z, assuming that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. Let $\mathbb{F}^t := \{\mathcal{F}^t_s\}_{s \geq 0}$ be the completion with the null sets of $\mathcal{F}$ of the sigma-algebra generated by the increments $(Z_r - Z_t)$ for $t \leq r \leq s \vee t$. We define an $\mathbb{F}$-Brownian motion $\mathcal{B}(\cdot)$ in $\mathbb{R}^p$ (with $p \geq 1$). Let $0 \leq t \leq T$, the following system of controlled SDEs in $\mathbb{R}^d$ is considered

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))d\mathcal{B}(s) & s \in (t,T) \\ X(t) = x. \end{cases} \tag{3.3.1}$$

where $u \in \mathcal{U}$ set of $\mathbb{F}$-progressively measurable processes with values in $U \subset \mathbb{R}^m$ $(m \geq 1)$. Along the whole chapter the set U is assumed to be compact (i.e. (H_U) is satisfied). We also recall that under assumptions (H_b) and (H_σ) on the dynamics, denoted by $X^u_{t,x}(\cdot)$ the strong solution of (3.3.1) associated with the control $u \in \mathcal{U}$, the classical estimates of Proposition 2.1.1 hold.

Let $\mathcal{T} \subseteq \mathbb{R}^d$ be a target set and $\mathcal{K} \subseteq \mathbb{R}^d$ a set of state constraints such that

$(H_\mathcal{T})$ $\qquad \mathcal{T} \subseteq \mathbb{R}^d$ is a nonempty and closed set.

$(H_\mathcal{K})$ $\qquad \mathcal{K} \subseteq \mathbb{R}^d$ is a nonempty and closed set.

As mentioned in the introduction we aim to characterize and compute the state constrained backward reachable set

$$\mathcal{R}^{\mathcal{T},\mathcal{K}}_t := \Big\{ x \in \mathbb{R}^d : \exists u \in \mathcal{U} \text{ such that } \Big(X^u_{t,x}(T) \in \mathcal{T} \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \Big) \text{ a.s. } \Big\}. \tag{3.3.2}$$

In what follows we will apply a "level set" approach. The bases of this approach were introduced by Osher and Sethian in [147] for fronts propagation problems in a deterministic framework. The main idea contained in this work is that it is possible to describe a curve, the curve representing the front in that case, as a level set of a suitable continuous function. Thanks to this observation, in [147] the propagation of the front is described by the 0-level set of a function that is characterized as the unique solution of an evolutionary PDE. In the deterministic framework (the case $\sigma \equiv 0$) the same idea has been applied with success by many authors in the later years. Among them we mention [98] for rendez-vous problems and [138, 139] for minimum time problems. We are in particular interested in state constrained reachability problems and in this case the level set method has been applied in [47] and [121].

We start with a short discussion on the application of the level set approach for solving stochastic $(\sigma \not\equiv 0)$ reachability problems in the unconstrained case $\mathcal{K} = \mathbb{R}^d$. Let us define a function $g_\mathcal{T}$ such that

$(H_{g_\mathcal{T}})$ $\qquad g_\mathcal{T} : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz continuous fuction,

$$g_\mathcal{T}(x) \geq 0 \quad \text{and} \quad g_\mathcal{T}(x) = 0 \Leftrightarrow x \in \mathcal{T}.$$

We consider the following stochastic optimal control problem:

$$v_\mathcal{T}(t,x) := \inf_{u \in \mathcal{U}} \mathbb{E}\Big[g_\mathcal{T}(X^u_{t,x}(T)) \Big]. \tag{3.3.3}$$

It is not difficult to prove that, as soon as the infimum in (3.3.3) is attained, one has $\mathcal{R}^{\mathcal{T},\mathcal{K}}_t = \{x \in \mathbb{R}^d : v_\mathcal{T}(x) = 0\}$. We will call $v_\mathcal{T}$ the *level set function*. Problem (3.3.3)

is a stochastic optimal control problem in a classical Mayer form, then all the results mentioned in Chapter 2 apply and $v_{\mathcal{T}}$ can be characterized as the unique viscosity solution of an HJB of the form (2.3.1). By the use of the available numerical methods for HJB equations, $v_{\mathcal{T}}$ can therefore be computed and, looking at its 0-level set, a numerical approximation of the backward reachable set is then obtained.
When state constraints are taken into account, that is if $\mathcal{K} \subset \mathbb{R}^d$, the suitable level set function to be considered is given by the following state constrained optimal control problem

$$v_{\mathcal{T},\mathcal{K}}(t,x) : \inf_{u\in\mathcal{U}} \left\{ \mathbb{E}\left[g_{\mathcal{T}}(X^u_{t,x}(T)) \right] : X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.} \right\}. \tag{3.3.4}$$

In absence of compatibility assumption between the dynamics and the set of state constraints, the characterization of the value function as the solution of the state constrained HJB equation becomes more complicated. We remand to Chapter 5 and the references therein for a more detailed discussion on this subject and for a presentation of the existing literature.
In order to overcome these difficulties and to avoid to deal with state constrained optimal control problems, in what follows we adapt the approach proposed in [47] to the stochastic context. The main idea is to manage the state constraints introducing an *exact penalization* term directly in the definition of the level set function. With this purpose we introduce another function $g_{\mathcal{K}}$ such that

$$\begin{aligned} & g_{\mathcal{K}} : \mathbb{R}^d \to \mathbb{R} \text{ is a Lipschitz continuous fuction,} \\ & g_{\mathcal{K}}(x) \geq 0 \quad \text{and} \quad g_{\mathcal{K}}(x) = 0 \Leftrightarrow x \in \mathcal{K}. \end{aligned} \tag{$H_{g_{\mathcal{K}}}$}$$

Remark 3.3.1. We can observe that under assumptions ($H_{\mathcal{T}}$),($H_{\mathcal{K}}$) it is always possible to find functions $g_{\mathcal{T}}$ and $g_{\mathcal{K}}$ satisfying ($H_{g_{\mathcal{T}}}$) and ($H_{g_{\mathcal{K}}}$). Indeed it is sufficient to define

$$g_{\mathcal{T}}(x) := d^+_{\mathcal{T}}(x) \quad \text{and} \quad g_{\mathcal{K}}(X) := d^+_{\mathcal{K}}(x)$$

where $d^+_{\mathcal{T}}$ and $d^+_{\mathcal{K}}$ denote the positive Euclidean distance functions to the sets $\mathcal{T}$ and $\mathcal{K}$ respectively.

Let us consider the following optimal control problem in a "maximum form":

$$w_{\mathcal{T},\mathcal{K}}(t,x) := \inf_{u\in\mathcal{U}} \mathbb{E}\left[g_{\mathcal{T}}(X^u_{t,x}(T)) \vee \max_{s\in[t,T]} g_{\mathcal{K}}(X^u_{t,x}(s)) \right]. \tag{3.3.5}$$

We have the following result

Proposition 3.3.2. *Let assumptions (H_b),(H_σ),($H_{\mathcal{T}}$),($H_{\mathcal{K}}$),($H_{g_{\mathcal{T}}}$) and ($H_{g_{\mathcal{K}}}$) be satisfied and let the infimum in (3.3.5) be attained. Then for any $t \in [0,T]$*

$$x \in \mathcal{R}^{\mathcal{T},\mathcal{K}}_t \Leftrightarrow w_{\mathcal{T},\mathcal{K}}(t,x) = 0.$$

Proof. On one hand, from the definition of $\mathcal{R}^{\mathcal{T},\mathcal{K}}_t$ and assumptions ($H_{g_{\mathcal{T}}}$), ($H_{g_{\mathcal{K}}}$) it follows that if $x \in \mathcal{R}^{\mathcal{T},\mathcal{K}}_t$ there exists $u \in \mathcal{U}$ such that

$$\left(g_{\mathcal{T}}(X^u_{t,x}(T)) = 0 \text{ and } g_{\mathcal{K}}(X^u_{t,x}(s)) = 0, \ \forall s \in [t,T] \right) \quad \text{a.s. .}$$

Hence

$$g_{\mathcal{T}}(X^u_{t,x}(T)) \vee \max_{s\in[t,T]} g_{\mathcal{K}}(X^u_{t,x}(s)) = 0, \qquad \text{a.s.}$$

and, observing also that $w_{\mathcal{T},\mathcal{K}}$ is always positive, we get $w_{\mathcal{T},\mathcal{K}}(t,x) = 0$.
On the other hand let us assume that $w_{\mathcal{T},\mathcal{K}}(t,x) = 0$. Since, by hypothesis, the infimum in (3.3.5) is attained for some control $\bar{u} \in \mathcal{U}$, we have that for such a control

$$\mathbb{E}\left[g_{\mathcal{T}}(X^{\bar{u}}_{t,x}(T)) \vee \max_{s\in[t,T]} g_{\mathcal{K}}(X^{\bar{u}}_{t,x}(s))\right] = 0.$$

Because of the non negativity of the process inside the expectation, it follows that

$$g_{\mathcal{T}}(X^{\bar{u}}_{t,x}(T)) \vee \max_{s\in[t,T]} g_{\mathcal{K}}(X^{\bar{u}}_{t,x}(s)) = 0 \qquad \text{a.s.}$$

and then, thanks again to assumptions ($H_{g_{\mathcal{T}}}$) and ($H_{g_{\mathcal{K}}}$), we can conclude that

$$\left(X^u_{t,x}(T) \in \mathcal{T} \text{ and } X^u_{t,x}(s) \in \mathcal{K},\ \forall s \in [t,T]\right) \qquad \text{a.s. ,}$$

that is $x \in \mathcal{R}^{\mathcal{T},\mathcal{K}}_t$. □

Remark 3.3.3. As pointed out by the proof above, the existence of an optimal control for problem (3.3.5) plays an important role for proving the second implication. It is nowadays well known (see Section 2.2 in Chapter 2) that a sufficient condition that guarantees such a property (if a weak formulation of the optimal control problem is considered) is the following

$$\{(b, \sigma\sigma^T)(t,x,u), u \in U\} \quad \text{is a convex set .}$$

An existence result under strong formulation, for the case of a linear dynamics, will be discussed in Section 3.9 at the end of the chapter. In absence of the existence assumption the backward reachable set is not closed. In this case, choosing $g_{\mathcal{T}} = d^+_{\mathcal{T}}$ and $g_{\mathcal{K}} = d^+_{\mathcal{K}}$ the 0-level set of $w_{\mathcal{T},\mathcal{K}}$ is associated with the following weaker notion of state constrained backward reachable set:

$$\begin{aligned}
\widetilde{\mathcal{R}}^{\mathcal{T},\mathcal{K}}_t := \Big\{ & x \in \mathbb{R}^d : \forall \varepsilon > 0, \exists u_\varepsilon \in \mathcal{U} \text{ such that} \\
& \mathbb{E}[d^+_{\mathcal{T}}(X^{u_\varepsilon}_{t,x}(T))] \le \varepsilon \text{ and } \mathbb{E}[\max_{s\in[t,T]} d^+_{\mathcal{K}}(X^{u_\varepsilon}_{t,x}(s))] \le \varepsilon \Big\} \\
= \Big\{ & x \in \mathbb{R}^d : \forall \varepsilon > 0, \exists u_\varepsilon \in \mathcal{U},\ \mathbb{E}[d^+_{\mathcal{T}}(X^{u_\varepsilon}_{t,x}(T)) \vee \max_{s\in[t,T]} d^+_{\mathcal{K}}(X^{u_\varepsilon}_{t,x}(s))] \le \varepsilon \Big\},
\end{aligned}$$

where the second characterization of $\widetilde{\mathcal{R}}^{\mathcal{T},\mathcal{K}}_t$ is easily deduced from the following inequalities:

$$\begin{aligned}
\mathbb{E}\left[d^+_{\mathcal{T}}(X^{u_\varepsilon}_{t,x}(T))\right] \vee \mathbb{E}\left[\max_{s\in[t,T]} d^+_{\mathcal{K}}(X^{u_\varepsilon}_{t,x}(s))\right] &\le \mathbb{E}\left[d^+_{\mathcal{T}}(X^{u_\varepsilon}_{t,x}(T)) \vee \max_{s\in[t,T]} d^+_{\mathcal{K}}(X^{u_\varepsilon}_{t,x}(s))\right] \\
&\le \mathbb{E}\left[d^+_{\mathcal{T}}(X^{u_\varepsilon}_{t,x}(T))\right] + \mathbb{E}\left[\max_{s\in[t,T]} d^+_{\mathcal{K}}(X^{u_\varepsilon}_{t,x}(s))\right].
\end{aligned}$$

The main advantage of reformulating the state constrained reachability problem using (3.3.5) is that the presence of state constraints does not represent a problem any more. In fact (3.3.5) is an unconstrained optimal control problem, i.e. the infimum is taken over the whole set of controls $\mathcal{U}$, and the state constraints only appear in the exact penalization term "$\max_{s\in[t,T]} g_{\mathcal{K}}(X^u_{t,x}(s))$". The term "exact" emphasizes the difference with the usual penalty method used in optimization, where the solution of the constrained problem is obtained as result of a limit procedure.

Remark 3.3.4. The choice of the level set function $w_{\mathcal{T},\mathcal{K}}$ is not, of course, the unique to guarantee the equivalence in Proposition 3.3.2. Indeed another possible choice is

$$\widetilde{w}_{\mathcal{T},\mathcal{K}}(t,x) := \inf_{u\in\mathcal{U}} \mathbb{E}\Big[g_{\mathcal{T}}(X^u_{t,x}(T)) + \int_t^T g_{\mathcal{K}}(X^u_{t,x}(s))ds\Big].$$

From the theoretical point of view the study of this optimal control problem is easier. It is in fact an unconstrained optimal control problem in a standard Bolza form and by classical dynamic programming arguments (see Chapter 2) $\widetilde{w}_{\mathcal{T},\mathcal{K}}$ can be characterized as the unique viscosity solution of the following HJB equation:

$$\begin{cases} -\partial_t w + H(t,x,D_x w, D_x^2 w) = 0 & t\in[0,T), x\in\mathbb{R}^d \\ w(T,x) = g_{\mathcal{T}}(x) & x\in\mathbb{R}^d \end{cases}$$

with

$$H(t,x,p,Q) := \sup_{u\in U}\Big\{ -b(t,x,u)\cdot p - \frac{1}{2}Tr[\sigma\sigma^T(t,x,u)Q]\Big\} - g_{\mathcal{K}}(x).$$

By the way it has been shown in the deterministic case that the "maximum problem" (3.3.5) leads to better numerical results (we refer to [46] for a discussion on this fact). For this reason in what follows our aim will be to propose a complete study of the extension to the stochastic setting of the approach based on the "maximum exact penalization".

Remark 3.3.5. We point out that in the application of the level set approach no further difficulties arise taking into account the case of moving targets and constraints. Indeed, in order to include also this case in our model it will be sufficient to consider functions $g_{\mathcal{T}}$ and $g_{\mathcal{K}}$ depending on t and the level set function $w_{\mathcal{T},\mathcal{K}}$ (resp. $\widetilde{w}_{\mathcal{T},\mathcal{K}}$) would be

$$w_{\mathcal{T},\mathcal{K}}(t,x) = \inf_{u\in\mathcal{U}} \mathbb{E}\Big[g_{\mathcal{T}}(X^u_{t,x}(T),T) \vee \max_{s\in[t,T]} g_{\mathcal{K}}(X^u_{t,x}(s),s)\Big]$$

$$\left(\text{resp. } \widetilde{w}_{\mathcal{T},\mathcal{K}}(t,x) := \inf_{u\in\mathcal{U}} \mathbb{E}\Big[g_{\mathcal{T}}(X^u_{t,x}(T),T) + \int_t^T g_{\mathcal{K}}(X^u_{t,x}(s),s)ds\Big]\right).$$

In conclusion, in view of Proposition 3.3.2 the reachable set $\mathcal{R}_t^{\mathcal{T},\mathcal{K}}$ can be characterized by means of the value function of a control problem with supremum cost in the form of (3.1.1), with functions g and ψ defined by: $g(x) := g_{\mathcal{K}}(x)$ and $\psi(x,y) := g_{\mathcal{T}}(x) \vee y$. For this reason in the rest of the chapter we will deal with the general formulation (3.1.1), coming back to the reachability problem in Section 3.8.

3.4 Stochastic optimal control problem with a maximum cost

As said in the previous section, we are now interested in stochastic optimal control problems with a cost function of the form

$$J(t,x,u) = \mathbb{E}\left[\psi\left(X^u_{t,x}(T), \max_{s\in[t,T]} g(X^u_{t,x}(s))\right)\right], \tag{3.4.1}$$

where $\psi : \mathbb{R}^{d+1} \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are two functions satisfying:

(H'_ψ) ψ is a bounded and Lipschitz continuous function;

(H_g) g is a Lipschitz continuous function.

We denote by L_ψ and L_g their Lipschitz constants. The value function associated with the cost (3.4.1) is defined by

$$v(t,x) = \inf_{u\in\mathcal{U}} J(t,x,u).$$

In the sequel, for any function φ defined from some set $Q \subset (0,\infty)\times\mathbb{R}^d$ into either $\mathbb{R}, \mathbb{R}^d$ or some space of matrices, we set

$$[\varphi]_1 := \sup_{(t,x)\neq(s,y)} \frac{|\varphi(t,x)-\varphi(s,y)|}{(|x-y|+|t-s|^{1/2})},$$

and

$$|\varphi|_1 := \|\varphi\|_\infty + [\varphi]_1.$$

The following assumptions on the coefficients b and σ will be required:

$$(H'_{b,\sigma}) \qquad \begin{cases} (i) & \sigma(\cdot,\cdot,\cdot), b(\cdot,\cdot,\cdot) \text{ are continuous on } [0,T]\times\mathbb{R}^d\times U; \\ (ii) & \exists M_b, M_\sigma \geq 0 \text{ independent on } u\in U \text{ such that} \\ & \qquad |b(\cdot,\cdot,u)|_1 \leq M_b, \quad |\sigma(\cdot,\cdot,u)|_1 \leq M_\sigma, \qquad \forall u \in U \end{cases}$$

Under assumptions $(H'_{b,\sigma})$, (H'_ψ) and (H_g), v is a Lipschitz continuous function in x and a $\frac{1}{2}$-Hölder continuous function in t (by the same arguments as in Proposition 3.4.1 below).

The main contributions to the study of this kind of problems can be found in [29] and [37] (see also [108] for the stationary case in the framework of classical solutions). In these works the dynamic programming techniques are applied on the L^p-approximation of the L^∞-cost functional in (3.4.1), using the approximation:

$$a \vee b \simeq (a^p + b^p)^{\frac{1}{p}} \qquad (\text{for } p \to \infty),$$

for any $a, b \geq 0$, where $a \vee b := \max(a,b)$. Then the HJ characterization for the original "maximum problem" is obtained as limit for $p \to \infty$. A fundamental hypothesis in order to apply this approach is the positivity of the functions involved. In our work, a direct derivation of a Dynamic Programming Principle gives an alternative and natural way for dealing with the running maximum cost problems under less restrictive assumptions. By this way, the optimal control problem associated to the cost functional (3.4.1) is connected to the solution of a HJB equation with oblique derivative boundary conditions. Here, the boundary conditions have to be understood in the viscosity sense (see [109]).

In order to characterize the function v as solution of an HJB equation the main tool is the well-known optimality principle (see Section 2.3). The particular non-Markovian structure of the cost functional (3.4.1) prohibits the direct use of the standard techniques. To avoid this difficulty it is classical to reformulate the problem by adding a new variable

$y \in \mathbb{R}$ that, roughly speaking, keeps the information of the running maximum. For this reason, we introduce an auxiliary value function ϑ defined on $[0,T]\times\mathbb{R}^d\times\mathbb{R}$ by:

$$\vartheta(t,x,y) := \inf_{u\in\mathcal{U}} \mathbb{E}\left[\psi\left(X^u_{t,x}(T),\ \max_{s\in[t,T]} g(X^u_{t,x}(s))\vee y\right)\right]. \tag{3.4.2}$$

The following property holds:

$$\vartheta(t,x,g(x)) = v(t,x), \tag{3.4.3}$$

so from now on, we will only work with the value function ϑ since the value of v can be obtained by (3.4.3) .

Proposition 3.4.1. *Under assumptions ($H'_{b,\sigma}$),(H_g) and (H'_ψ), ϑ is a Lipschitz continuous function in (x,y) uniformly with respect to t, and a $\frac{1}{2}$-Hölder continuous function in t. More precisely, there exists $L_\vartheta > 0$ (L_ϑ depends only on $M_b, M_\sigma, L_\psi, L_g$ and T) such that:*

$$|\vartheta(t,x,y)-\vartheta(t',x',y')| \le L_\vartheta(|x-x'|+|y-y'|+(1+|x|)|t-t'|^{1/2}),$$

for all $(x,y),(x',y')\in\mathbb{R}^{d+1}$ and for any $t\le t'\in[0,T]$.

Proof. Let be $t\le t'\le T$ and $x,x'\in\mathbb{R}^d$. Notice that the following property holds for the maximum

$$|(a\vee b)-(c\vee d)| \le |a-c|\vee|b-d|\,.$$

Then, the inequalities (2.1.2) yield to:

$$\left|\vartheta(t,x,y)-\vartheta(t,x',y)\right| \le K\sup_{u\in\mathcal{U}}\mathbb{E}\left[\sup_{s\in[t,T]}\left|X^u_{t,x}(s)-X^u_{t,x'}(s)\right|\right] \le KC|x-x'|$$

where $K := L_\psi(L_g+1)$ and C is the constant appearing in Proposition 2.1.1. In a similar way, we obtain

$$\begin{aligned}\left|\vartheta(t,x,y)-\vartheta(t',x,y)\right| \le L_\psi\sup_{u\in\mathcal{U}}\mathbb{E}\Bigg[&\left|X^u_{t,x}(T)-X^u_{t',x}(T)\right| + L_g\sup_{s\in[t,t']}\left|X^u_{t,x}(s)-x\right| +\\ &+ L_g\sup_{s\in[t',T]}\left|X^u_{t',x}(s)-X^u_{t',X^u_{t,x}(t')}(s)\right|\Bigg]\\ \le{}& K'(1+|x|)\left|t-t'\right|^{1/2}\end{aligned}$$

for a positive constant $K'>0$ that depends only on L_ψ, L_g and C. The L_ψ-Lipschitz behavior in the variable y is immediate. We conclude then the result with $L_\vartheta = K\vee K'\vee L_\psi$. □

3.4.1 Link with lookback options in finance

Another interest for computing expectation of supremum cost functionals is the study of lookback options in Finance. The value of such an option is typically of the form

$$\mathbb{E}\left[e^{-\int_t^T r(s)ds}\psi\big(X_{t,x}(T),\ \max_{s\in[t,T]} g(X_{t,x}(s))\big)\right],$$

where $X_{t,x}(\cdot)$ (the "asset") is a solution of a one-dimensional SDE (3.3.1), $g(x) = x$, $r(\cdot)$ is the interest rate, and ψ is the payoff function. Here the option value depends not only on the value of the asset at time T but also on all the values taken between times t and T. A detailed description of this model can be found in [171]. Typical payoff functions are $\psi(x, y) = y-x$ (lookback floating strike put), $\psi(x, y) = \max(y-E, 0)$ (fixed strike lookback call), $\psi(x, y) = \max(\min(y, E) - x, 0)$ (lookback limited-risk put), etc., see [171, 30] (see also [29] for other examples and related american lookback options).

3.5 The Hamilton-Jacobi-Bellman equation

The aim of this section is to characterize the value function ϑ defined by (3.4.2) as the unique viscosity solution of an HJB equation. Stochastic optimal control problems with running maximum cost in the viscosity solutions framework have been studied in [29, 37]. The arguments developed in that papers are based on the approximation technique of the L^∞-norm. Here, we derive the HJB equation directly without using any approximation. In all the sequel, we will use the abbreviation *a.e.* for *almost every*.

3.5.1 Dynamic Programming

Let us start defining the process

$$Y^u_{t,x,y}(\cdot) := \max_{s\in[t,\cdot]} g(X^u_{t,x}(s)) \vee y. \tag{3.5.1}$$

The optimal control problem (3.4.2) can be re-written as

$$\vartheta(t, x, y) := \inf_{u\in\mathcal{U}} \mathbb{E}\left[\psi\left(X^u_{t,x}(T), Y^u_{t,x,y}(T)\right)\right]. \tag{3.5.2}$$

We point out that that (3.5.2) can be seen as a Mayer's problem in the augmented state space.

In order to characterize ϑ as a solution of an HJB equation the first step is to obtain a Bellman's principle. It is stated in the following theorem:

Theorem 3.5.1 (DPP)**.** *Under hypothesis* $(H'_{b,\sigma})$,(H'_ψ) *and* (H_g), *for all* $(t, x) \in [0, T)\times \mathbb{R}^d$ *and all family of* $\mathbb{F}^t$*-stopping times* $\{\theta^u, u \in \mathcal{U}\}$ *with values in* $[t, T]$*:*

$$\vartheta(t, x, y) = \inf_{u\in\mathcal{U}} \mathbb{E}\Big[\vartheta\big(\theta^u, X^u_{t,x}(\theta^u), Y^u_{t,x,y}(\theta^u)\big)\Big]. \tag{3.5.3}$$

A proof of Theorem 3.5.1 can be obtained by adapting the same arguments developed by Bouchard and Touzi in [63], thanks to the fact that the couple of variables $(X^u_{t,x}(\cdot), Y^u_{t,x,y}(\cdot))$ satisfies the following fundamental property

$$\begin{pmatrix} X^u_{t,x}(s) \\ Y^u_{t,x,y}(s) \end{pmatrix} = \begin{pmatrix} X^u_{\theta,X^u_{t,x}(\theta)}(s) \\ Y^u_{\theta,X^u_{t,x}(\theta),Y^u_{t,x,y}(\theta)}(s) \end{pmatrix} \qquad \text{a.s.}$$

for any stopping time θ with $t \le \theta \le s \le T$. In our case the proof is even simpler than the one of [63] thanks to the uniform continuity of ϑ (Proposition 3.4.1).

3.5.2 Hamilton-Jacobi-Bellman equation

Theorem 3.5.1 is the main tool for proving next result that characterizes ϑ as a solution, in viscosity sense, of a HJB equation with oblique derivative boundary conditions. Set

$$\overline{D} := \Big\{(x,y) \in \mathbb{R}^{d+1} : y \geq g(x)\Big\} = \text{Epigraph}(g), \tag{3.5.4}$$

where D is the interior of $\overline{D}$.

Theorem 3.5.2. *Under assumptions* $(H'_{b,\sigma})$,(H'_{ψ}) *and* (H_g), ϑ *is a continuous bounded viscosity solution of the following HJB equation*

$$-\partial_t \vartheta + H(t,x,D_x\vartheta, D_x^2\vartheta) = 0 \quad \textit{in } [0,T)\times D \tag{3.5.5a}$$

$$-\partial_y \vartheta = 0 \quad \textit{on } [0,T)\times \partial D \tag{3.5.5b}$$

$$\vartheta(T,x,y) = \psi(x,y) \quad \textit{in } \overline{D} \tag{3.5.5c}$$

with

$$H(t,x,p,Q) := \sup_{u\in U}\left\{-b(t,x,u)p - \frac{1}{2}Tr[\sigma\sigma^T](t,x,u)Q\right\}. \tag{3.5.6}$$

Before starting the proof we recall the definition of viscosity solution for problem (3.5.5) (see [84] and the references therein for a complete discussion on weak boundary conditions).

Definition 3.5.3. An USC (resp. LSC) function ϑ on $[0,T]\times\overline{D}$ is a viscosity sub-solution (resp. super-solution) of (3.5.5), if for each function $\varphi \in C^{1,2}([0,T]\times\overline{D})$, at each maximum (resp. minimum) point (t,x,y) of $\vartheta-\varphi$ the following inequalities hold

$$\begin{cases} -\partial_t\varphi + H(t,x,D_x\varphi,D_x^2\varphi) \leq 0 & \text{in } [0,T)\times D \\ \min\big(-\partial_t\varphi + H(t,x,D_x\varphi,D_x^2\varphi), -\partial_y\varphi\big) \leq 0 & \text{on } [0,T)\times\partial D \\ \min\big(\vartheta-\psi, -\partial_t\varphi + H(t,x,D_x\varphi,D_x^2\varphi), -\partial_y\varphi\big) \leq 0 & \text{on } \{T\}\times\partial D \\ \min\big(\vartheta-\psi, -\partial_t\varphi + H(t,x,D_x\varphi,D_x^2\varphi)\big) \leq 0 & \text{on } \{T\}\times D. \end{cases}$$

(resp.

$$\begin{cases} -\partial_t\varphi + H(t,x,D_x\varphi,D_x^2\varphi) \geq 0 & \text{in } [0,T)\times D \\ \max\big(-\partial_t\varphi + H(t,x,D_x\varphi,D_x^2\varphi), -\partial_y\varphi\big) \geq 0 & \text{on } [0,T)\times\partial D \\ \max\big(\vartheta-\psi, -\partial_t\varphi + H(t,x,D_x\varphi,D_x^2\varphi), -\partial_y\varphi\big) \geq 0 & \text{on } \{T\}\times\partial D \\ \max\big(\vartheta-\psi, -\partial_t\varphi + H(t,x,D_x\varphi,D_x^2\varphi)\big) \geq 0 & \text{on } \{T\}\times D). \end{cases}$$

Finally a continuous function ϑ is a viscosity solution of (3.5.5) if it is both a sub- and a super-solution.

Proof of Theorem 3.5.2. First, from the definition of ϑ and thanks to its regularity (see Proposition 3.4.1), we obtain easily that $\vartheta(T,x,y) = \psi(x,y)$.

Now, we check that ϑ is a viscosity sub-solution. Let $\varphi \in C^{1,2}([0,T]\times\overline{D})$ such that $\vartheta-\varphi$ attains a maximum at point $(\bar t,\bar x,\bar y) \in [0,T)\times\overline{D}$. Without loss of generality we can always assume that $(\bar t,\bar x,\bar y)$ is a strict local maximum point (let us say in a ball of radius $r>0$ centered in $(\bar t,\bar x,\bar y)$) and $\vartheta(\bar t,\bar x,\bar y) = \varphi(\bar t,\bar x,\bar y)$. Thanks to Theorem 3.5.1, for any $u \in \mathcal{U}$ and for any sufficiently small stopping time $\theta = \theta^u$, we have:

$$\begin{aligned} \varphi(\bar t,\bar x,\bar y) = \vartheta(\bar t,\bar x,\bar y) &\leq \mathbb{E}\Big[\vartheta\big(\theta, X^u_{\bar t,\bar x}(\theta), Y^u_{\bar t,\bar x,\bar y}(\theta)\big)\Big] \\ &\leq \mathbb{E}\Big[\varphi\big(\theta, X^u_{\bar t,\bar x}(\theta), Y^u_{\bar t,\bar x,\bar y}(\theta)\big)\Big]. \end{aligned} \tag{3.5.7}$$

Two cases will be considered depending on if the point $(\bar{x}, \bar{y})$ belongs or not to the boundary of D.

— Case 1: $g(\bar{x}) < \bar{y}$. Consider a constant control $u(s) \equiv u \in U$. From the continuity of g and the a.s. continuity of the sample paths it follows that, for a.e. $\omega \in \Omega$, there exists $\bar{s}(\omega) > \bar{t}$ such that $g(X^u_{\bar{t},\bar{x}}(s)) < \bar{y}$ if $s \in [\bar{t}, \bar{s}(\omega))$. Let be $h > 0$, and let $\bar{\theta}$ be a the following stopping time:

$$\bar{\theta} := \inf\left\{s > \bar{t} | (X^u_{\bar{t},\bar{x}}(s), Y^u_{\bar{t},\bar{x},\bar{y}}(s)) \notin B((\bar{x}, \bar{y}), r)\right\} \wedge (\bar{t} + h) \wedge \inf\left\{s > \bar{t} | g(X^u_{\bar{t},\bar{x}}(s)) \geq \bar{y}\right\}, \tag{3.5.8}$$

(here and in the sequel $B(x, r)$ denotes the ball of radius $r > 0$ centered at x). One can easily observe that a.s. $Y^u_{\bar{t},\bar{x},\bar{y}}(\bar{\theta}) = \bar{y}$, then by (3.5.7)

$$\varphi(\bar{t}, \bar{x}, \bar{y}) \leq \mathbb{E}\left[\varphi(\bar{\theta}, X^u_{\bar{t},\bar{x}}(\bar{\theta}), \bar{y})\right] \qquad \forall u \in U.$$

By applying the Ito's formula, and thanks to the smoothness of φ, we get:

$$\begin{aligned}\mathbb{E}\bigg[\int_{\bar{t}}^{\bar{\theta}} &-\partial_t\varphi(s, X^u_{\bar{t},\bar{x}}(s), \bar{y}) - b(s, X^u_{\bar{t},\bar{x}}(s), u) D_x\varphi(s, X^u_{\bar{t},\bar{x}}(s), \bar{y}) \\ &- \frac{1}{2}Tr[\sigma\sigma^T(s, X^u_{\bar{t},\bar{x}}(s), u) D^2_x\varphi(s, X^u_{\bar{t},\bar{x}}(s), \bar{y})]ds\bigg] \leq 0.\end{aligned}$$

Note that the stopping times

$$\inf\left\{s > \bar{t} | (X^u_{\bar{t},\bar{x}}(s), Y^u_{\bar{t},\bar{x},\bar{y}}(s)) \notin B_r(\bar{x}, \bar{y})\right\} \quad \text{and} \quad \inf\left\{s > \bar{t} \mid g(X^u_{\bar{t},\bar{x}}(s)) \geq \bar{y}\right\}$$

are a.s. strictly greater than $\bar{t}$, then for h sufficiently small in (3.5.8) one obtains $\bar{\theta} = \bar{t} + h$. One has

$$\begin{aligned}\mathbb{E}\bigg[\frac{1}{h}\int_{\bar{t}}^{\bar{\theta}} &-\partial_t\varphi(s, X^u_{\bar{t},\bar{x}}(s), \bar{y}) - b(s, X^u_{\bar{t},\bar{x}}(s), u) D_x\varphi(s, X^u_{\bar{t},\bar{x}}(s), \bar{y}) \\ &- \frac{1}{2}Tr[\sigma\sigma^T(s, X^u_{\bar{t},\bar{x}}(s), u) D^2_x\varphi(s, X^u_{\bar{t},\bar{x}}(s), \bar{y})]ds\bigg] \leq 0.\end{aligned}$$

By the dominate convergence theorem and the mean value theorem, letting h going to 0, it follows

$$-\partial_t\varphi(\bar{t}, \bar{x}, \bar{y}) - b(\bar{t}, \bar{x}, u) D_x\varphi(\bar{t}, \bar{x}, \bar{y}) - \frac{1}{2}Tr[\sigma\sigma^T(\bar{t}, \bar{x}, u) D^2_x\varphi(\bar{t}, \bar{x}, \bar{y})] \leq 0, \qquad \forall u \in U,$$

and finally:

$$-\partial_t\varphi(\bar{t}, \bar{x}, \bar{y}) + \sup_{u \in U}\left\{-b(\bar{t}, \bar{x}, u) D_x\varphi(\bar{t}, \bar{x}, \bar{y}) - \frac{1}{2}Tr[\sigma\sigma^T(\bar{t}, \bar{x}, u) D^2_x\varphi(\bar{t}, \bar{x}, \bar{y})]\right\} \leq 0.$$

— Case 2: $g(\bar{x}) = \bar{y}$. Assume that $-\partial_y\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, otherwise the conclusion is straightforward. Consider a constant control $u(s) \equiv u \in U$. Thanks to the continuity of the sample paths and the smoothness of φ, for a.e. ω there is a time $\bar{s}(\omega) > \bar{t}$ and $\eta > 0$ such that:

$$\varphi(s, X^u_{\bar{t},\bar{x}}(s), y) \leq \varphi(s, X^u_{\bar{t},\bar{x}}(s), \bar{y}) \qquad \forall s \in [\bar{t}, \bar{s}], \;\; y \in [\bar{y}, \bar{y} + \eta).$$

Let $\bar\theta$ be the stopping time given by:

$$\bar\theta := \inf\Big\{s>\bar t \,|\, (X^u_{\bar t,\bar x}(s), Y^u_{\bar t,\bar x,\bar y}(s)) \notin B((\bar x,\bar y),r)\Big\} \wedge \inf\Big\{s>\bar t \,|\, g(X^u_{\bar t,\bar x}(s)) \notin [\bar y,\bar y+\eta)\Big\}$$
$$\wedge \inf\Big\{s>\bar t \,|\, \partial_y\varphi(s,X^u_{\bar t,\bar x}(s),\bar y)\geq 0\Big\} \wedge (\bar t+h)\,.$$

By (3.5.7) one has

$$\varphi(\bar t,\bar x,\bar y)\leq \mathbb{E}\Big[\varphi(\bar\theta, X^u_{\bar t,\bar x}(\bar\theta),\bar y)\Big],$$

which implies (as we have seen for Case 1):

$$-\partial_t\varphi(\bar t,\bar x,\bar y)+\sup_{u\in U}\Big\{-b(\bar t,\bar x,u)D\varphi(\bar t,\bar x,\bar y)-\frac{1}{2}Tr[\sigma\sigma^T(\bar t,\bar x,u)D^2\varphi(\bar t,\bar x,\bar y)]\Big\}\leq 0.$$

In conclusion at point $(\bar t,\bar x,\bar y)\in[0,T)\times\partial D$ one has

$$\min\Big(-\partial_t\varphi+\sup_{u\in U}\Big\{-b(\bar t,\bar x,u)D\varphi-\frac{1}{2}Tr[\sigma\sigma^T](\bar t,\bar x,u)D^2\varphi\Big\},-\partial_y\varphi\Big)\leq 0,$$

and ϑ is a viscosity sub-solution of equation (3.5.5).

It remains to prove that ϑ is a viscosity super-solution of (3.5.5). Let $\varphi\in C^{1,2}([0,T]\times\overline{D})$ be such that $\vartheta-\varphi$ attains a minimum at point $(\bar t,\bar x,\bar y)\in[0,T)\times\overline{D}$. Without loss of generality we can always assume that $(\bar t,\bar x,\bar y)$ is a strict local minimum point in a ball $B((\bar t,\bar x,\bar y),r)$ and $\vartheta(\bar t,\bar x,\bar y)=\varphi(\bar t,\bar x,\bar y)$. We consider again the two cases:
— Case 1: $g(\bar x)<\bar y$. We assume by contradiction that

$$-\partial_t\varphi(\bar t,\bar x,\bar y)+H(\bar t,\bar x,D_x\varphi(\bar t,\bar x,\bar y),D_x^2\varphi(\bar t,\bar x,\bar y))<0.$$

By using continuity arguments we can also state that

$$-\partial_t\varphi(\cdot,\cdot,\bar y)+H(\cdot,\cdot,D_x\varphi(\cdot,\cdot,\bar y),D^2_x\varphi(\cdot,\cdot,\bar y))\leq 0 \tag{3.5.9}$$

in a neighborhood $B((\bar t,\bar x),r_1)$ of $(\bar t,\bar x)$ for some $r_1>0$. Moreover, thanks to the continuity of g, if $\bar y-g(\bar x)=:\rho>0$ we can define $r_2>0$ such that

$$\max_{x\in B(\bar x,r_2)} g(x)-g(\bar x)\leq\frac{\rho}{2}$$

and we have

$$\max_{x\in B(\bar x,r_2)} g(x)\vee\bar y=\bar y.$$

For any $u\in\mathcal{U}$ we define the stopping time $\bar\theta^u$ as the first exit time of the process $(s,X^u_{\bar t,\bar x}(s))$ from the ball $B((\bar t,\bar x),R)$ for $R:=\min(r,r_1,r_2)>0$. The continuity of the sample paths implies that $(\bar\theta^u,X^u_{\bar t,\bar x}(\bar\theta^u))\in\partial B((\bar t,\bar x),R)$ a.s.. Being $(\bar t,\bar x,\bar y)$ a strict minimum point on has

$$\min_{\partial B((\bar t,\bar x),R)}(\vartheta-\varphi)=:\eta>0$$

and then a.s.

$$\vartheta(\bar\theta^u,X^u_{\bar t,\bar x}(\bar\theta^u),\bar y)\geq\varphi(\bar\theta^u,X^u_{\bar t,\bar x}(\bar\theta^u),\bar y)+\eta.$$

Applying the Ito's formula and taking the expectation we get

$$
\begin{aligned}
&\varphi(\bar t,\bar x,\bar y) - \mathbb{E}\Big[\varphi(\bar\theta^u, X^u_{\bar t,\bar x}(\bar\theta^u),\bar y)\Big] \\
&= \mathbb{E}\Big[\int_{\bar t}^{\bar\theta^u} -\partial_t\varphi(s, X^u_{\bar t,\bar x}(s),\bar y) - b(s, X^u_{\bar t,\bar x}(s), u(s))D_x\varphi(s, X^u_{\bar t,\bar x}(s),\bar y) \\
&\quad - \frac{1}{2}Tr[\sigma\sigma^T(s, X^u_{\bar t,\bar x}(s), u(s))D^2_x\varphi(s, X^u_{\bar t,\bar x}(s),\bar y)]ds\Big] \le 0
\end{aligned}
\tag{3.5.10}
$$

that leads to

$$
\begin{aligned}
\vartheta(\bar t,\bar x,\bar y) = \varphi(\bar t,\bar x,\bar y) &\le \mathbb{E}\Big[\varphi(\bar\theta^u, X^u_{\bar t,\bar x}(\bar\theta^u),\bar y)\Big] \le \mathbb{E}\Big[\vartheta(\bar\theta^u, X^u_{\bar t,\bar x}(\bar\theta^u),\bar y)\Big] - \eta \\
&= \mathbb{E}\Big[\vartheta(\bar\theta^u, X^u_{\bar t,\bar x}(\bar\theta^u), Y^u_{\bar t,\bar x,\bar y}(\bar\theta^u)\Big] - \eta.
\end{aligned}
$$

Since η does not depends on u and u is arbitrary, this contradicts Theorem 3.5.1.
— Case 2: $g(\bar x) = \bar y$. Assume by contradiction that

$$
-\partial_y\varphi(\bar t,\bar x,\bar y) < 0 \quad \text{and} \quad -\partial_t\varphi(\bar t,\bar x,\bar y) + H(\bar t,\bar x, D_x\varphi(\bar t,\bar x,\bar y), D^2_x\varphi(\bar t,\bar x,\bar y)) < 0.
$$

We can again define $r_1 > 0$ such that (3.5.9) is satisfied in $B((\bar t,\bar x), r_1)$. Moreover there exists $\tilde r_2 > 0$ such that

$$
\varphi(s,\xi,\zeta) \le \varphi(s,\xi,\zeta')
$$

for any $(s,\xi,\zeta),(s,\xi,\zeta') \in B((\bar t,\bar x,\bar y),\tilde r_2)$ such that $\zeta \le \zeta'$. For any $u \in \mathcal{U}$ we define the stopping time $\bar\theta^u$ as the first exit time of the process $(s, X^u_{\bar t,\bar x}(s), Y^u_{\bar t,\bar x,\bar y}(s))$ from the ball $B((\bar t,\bar x,\bar y),\tilde R)$ for $\tilde R := \min(r, r_1, \tilde r_2) > 0$. As for Case 1, we can still say that a.s.

$$
\vartheta(\bar\theta^u, X^u_{\bar t,\bar x}(\bar\theta^u), Y^u_{\bar t,\bar x,\bar y}(\bar\theta^u)) \ge \varphi(\bar\theta^u, X^u_{\bar t,\bar x}(\bar\theta^u), Y^u_{\bar t,\bar x,\bar y}(\bar\theta^u)) + \eta'.
$$

for some $\eta' > 0$ not depending on u. Therefore by using (3.5.10) and by observing that $Y^u_{\bar t,\bar x,\bar y}(s) \ge \bar y$ for any $s \ge 0$, we get

$$
\begin{aligned}
\vartheta(\bar t,\bar x,\bar y) = \varphi(\bar t,\bar x,\bar y) &\le \mathbb{E}\Big[\varphi(\bar\theta^u, X^u_{\bar t,\bar x}(\bar\theta^u),\bar y)\Big] \le \mathbb{E}\Big[\varphi(\bar\theta^u, X^u_{\bar t,\bar x}(\bar\theta^u), Y^u_{\bar t,\bar x,\bar y}(\bar\theta^u))\Big] \\
&\le \mathbb{E}\Big[\vartheta(\bar\theta^u, X^u_{\bar t,\bar x}(\bar\theta^u), Y^u_{\bar t,\bar x,\bar y}(\bar\theta^u)\Big] - \eta',
\end{aligned}
$$

the yields again to a contradiction of Theorem 3.5.1. □

3.5.3 Comparison principle

The section is concluded with a comparison principle for equation (3.5.5). There is a large literature dealing with Neumann-type or oblique derivative boundary conditions. We refer to [134, 34] for the first order case and to [113, 111] for second order equations. For dealing with this kind of problems some regularity assumption on the domain D is often required. In our case the definition of D is strictly connected with the choice of the function g that is, without further hypothesis, only Lipschitz continuous. The result below has the advantage of taking into account also this non smooth case.

Theorem 3.5.4. *Assume that* $(H'_{b,\sigma})$, (H'_ψ) *and* (H_g) *are satisfied. Let* $\overline{\vartheta}$ *(resp.* $\underline{\vartheta}$*) be an USC (resp. LSC) bounded viscosity sub-solution (resp. super-solution) of* (3.5.5)*. Then* $\overline{\vartheta} \le \underline{\vartheta}$ *on* $[0,T] \times \overline{D}$*.*

Such a comparison principle is proved in [93] and [57] for respectively elliptic and parabolic equations in bounded domains. The arguments extend the ones used in [113, 111] for the case when the domain D has a smooth boundary. We give the main steps of the proof in Section 3.10. Assertions of Theorems 3.5.2 and 3.5.4 lead to the following result:

Corollary 3.5.5. *Under assumptions* $(H'_{b,\sigma})$, (H'_{ψ}) *and* (H_g), *the value function* ϑ *is the unique continuous bounded viscosity solution of equation* (3.5.5) *on* $[0,T]\times\overline{D}$.

The uniqueness result is stated on $\overline{D}$, which means that the solution to the HJB equation coincides with the value function ϑ on $[0,T]\times\overline{D}$. Then ϑ is extended in a unique way on $[0,T]\times\mathbb{R}^d\times\mathbb{R}$ by: $\vartheta(t,x,y)=\vartheta(t,x,y\vee g(x))$.

3.6 Numerical approximation

We pass now to discuss the numerical aspects. First, we focus on general approximation schemes for HJB equations with oblique derivative boundary conditions as (3.5.5). Then, a convergence result is proved by using the general framework of Barles-Souganidis [36] based on the monotonicity, stability, consistency of the scheme (a precise definition of these notions, in the case of HJB equations with oblique derivative boundary condition, is given in section 3.6.1). Secondly, we focus on the fully discrete semi-Lagrangian scheme and we study some properties of the numerical approximation. Semi-Lagrangian schemes for second order HJB equations (in the general form of Markov chain approximations) have been first considered in the book [128]. In the framework of viscosity solutions, semi-Lagrangian schemes for first order HJB equations have been studied in [74]. Extensions to the second order case can be found in [141, 68, 87, 92]. In our context, the numerical scheme that will be studied couples the classical semi-Lagrangian scheme with an additional projection step on the boundary. For this particular scheme, we derive in the next section a rate of convergence, generalizing in this way the results already known for general HJB equations without boundary conditions.

Let $N\geq 1$ be an integer (number of time steps), and let

$$h:=\frac{T}{N},\quad \text{and}\quad t_n:=nh,$$

for $n=0,\ldots,N$. Let $\Delta x=(\Delta x_1,\ldots,\Delta x_d)\in(\mathbb{R}^*_+)^d$ be a mesh step in $\mathbb{R}^d$, $\Delta y>0$ be a mesh step in $\mathbb{R}$, and $\rho:=(h,\Delta x,\Delta y)$ be a set of mesh steps (in time and space).

For a given ρ, consider the corresponding space grid

$$\mathcal{G}_\rho:=\Big\{(x_i,y_j)=(i\Delta x,j\Delta y),\text{for }(i,j)\in\mathbb{Z}^d\times\mathbb{Z}\Big\}.$$

where for $i\in\mathbb{Z}^d$, $i\Delta x:=(i_1\Delta x_1,\cdots,i_N\Delta x_N)$. For any $x\in\mathbb{R}^d$, let $j_x\in\mathbb{Z}$ be the upper integer part of $\frac{g(x)}{\Delta y}$, i.e,

$$j_x:=\min\big\{j\in\mathbb{Z},\ j\Delta y\geq g(x)\big\}.$$

Consider a *projection* operator (along the direction e_y) $\Pi^{\mathcal{G}_\rho}$ defined as follows:

$$\Pi^{\mathcal{G}_\rho}(\Phi)(t_n,x_i,y_j):=\left\{\begin{array}{ll}\Phi(t_n,x_i,y_j), & \forall j\geq j_{x_i}, i\in\mathbb{Z}^d,\\ \Phi(t_n,x_i,y_{j_{x_i}}), & \forall j<j_{x_i}, i\in\mathbb{Z}^d,\end{array}\right.$$

for any $\Phi \in C([0,T]\times\mathbb{R}^d\times\mathbb{R},\mathbb{R})$.

We aim at computing an approximation $W^n_{i,j}$ of the value function $\vartheta(t_n,x_i,y_j)$, on the grid $\mathcal{G}_\rho$. By W^ρ, we will denote the interpolation of the values $W^n_{i,j}$ at (t_n,x_i,y_j). The values $W^n_{i,j}$ are defined recusively as follows:

General scheme (GS)

1) Define $W^N_{i,j} := \psi(x_i, y_j \vee g(x_i)), \forall i,j$.

2) For $n = N,\dots,1$, the value $W^{n-1}_{i,j}$ is obtained as solution of:

$$S^\rho(t_{n-1},x_i,y_j,W^{n-1}_{i,j},\Pi^{\mathcal{G}_\rho}(W^\rho)) = 0, \quad \forall i,j \text{ with } g(x_i)\le y_j, \tag{3.6.1}$$

$$W^{n-1}_{i,j} := W^{n-1}_{i,j_{x_i}}, \quad \forall i,j \text{ with } g(x_i) > y_j, \tag{3.6.2}$$

where $S^\rho : [0,T]\times\overline{D}\times\mathbb{R}\times C([0,T]\times\mathbb{R}^d\times\mathbb{R},\mathbb{R})\to\mathbb{R}$ is a scheme operator.

Typical operator S^ρ can be obtained by using an explicit or implicit finite difference method on (3.5.5a) (see [51, 50, 129]), or a semi-Lagrangian (SL) scheme ([141, 68, 92]). In (3.6.1), the value of $W^{n-1}_{i,j}$ may depend on the whole function W^ρ (i.e., all the values $W^k_{i,j}$ for $k = 0,\cdots,N$). Of course, in case of an explicit time discretization, the step (3.6.1) could be re-written as:

$$S^\rho(t_{n-1},x_i,y_j,W^{n-1}_{i,j},\Pi^{\mathcal{G}_\rho}(W^\rho(t_n,\cdot,\cdot))) = 0, \quad \forall i,j \text{ with } g(x_i)\le y_j.$$

However, the formulation (3.6.1) is more general and includes different kind of time-discretization like Euler implicit scheme or θ-methods.

The main idea of the numerical method described here above is to mix the use of a standard scheme for (3.5.5a), together with a "projection step" on ∂D in order to approximate the homogeneous oblique derivative boundary condition (3.5.5b). Let us point out that a similar method was introduced in [30] for the case $g(x)\equiv|x|$. However, the general case with possibly nonlinear function g requires some further attention on the boundary.

3.6.1 A general convergence result

In this part we closely follow the arguments of Barles and Souganidis [36], using consistency, stability and monotonicity arguments. For this, we assume that the following hypotheses are considered:

(H_S) (stability) For any ρ, there exists a solution W^ρ such that

$$|W^\rho(t,x,y)|\le M, \quad \text{on} \quad [0,T]\times\overline{D};$$

for some positive constant M independent on ρ;

(H_C) (consistency) The scheme S^ρ is *consistent* with respect to equation (3.5.5a) in $[0,T]\times\overline{D}$, that is, for all $(t,x,y)\in[0,T]\times\overline{D}$ and for every $\Phi\in C^{1,2}([0,T]\times\overline{D})$,

$$\lim_{\substack{\rho\to0\\ [0,T]\times\overline{D}\ni(s,\xi,\gamma)\to(t,x,y)\\ \zeta\to0}} S^\rho(s,\xi,\gamma,\Phi(s,\xi,\gamma)+\zeta,\Phi+\zeta) = -\partial_t\Phi + H(t,x,D_x\Phi,D^2_x\Phi).$$

(H_M) (monotonicity) For any ρ, for $r \in \mathbb{R}$, $(t,x,y) \in [0,T] \times \overline{D}$, $S^\rho(t,x,y,r,\Phi)$ depends only on the values of Φ in a neighborhood $B_{\eta(\rho)}(t,x,y)$ of (t,x,y), with $\eta(\rho) \geq 0$ such that $\eta(\rho) \xrightarrow{\rho \to 0} 0$. Moreover, for any Φ_1, Φ_2 functions on $[0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, such that $\Phi_1 \geq \Phi_2$ on $B_{\eta(\rho)}((t,x,y))$, it holds:

$$S^\rho(t,x,y,r,\Phi_1) \leq S^\rho(t,x,y,r,\Phi_2).$$

Let us point out that the monotonicity and the consistency are required here only for the operator scheme S^ρ that corresponds to the discretization of the equation $-\partial_t \vartheta + H(t,x,D\vartheta,D^2\vartheta) = 0$.

Notice also that the monotonicity assumption (H_M) is slightly different from the classical one usualy required for general HJB equations. The reason for this comes from the fact that the operator scheme S^ρ is defined on $\overline{D}$ and may use some values of the function W^ρ outside the domain D that are obtained by oblique projection through the projection step. Our monotonicity assumption requires local dependency on the values of Φ in a neighborhood of the point under consideration, then a comparison of the scheme values is requested for any two functions Φ_1 and Φ_2 such that $\Phi_1 \geq \Phi_2$ only on this neighborhood. The proof of Theorem 3.6.1 highlights what this requirement is needed for.

In the next section, we will check that assumptions (H_S),(H_C) and (H_M) are well satisfied by the semi-Lagrangian scheme. Now, we state the convergence result for any general scheme satisfying (H_S),(H_C) and (H_M).

Theorem 3.6.1. *Assume $(H'_{b,\sigma})$, (H'_ψ) and (H_g). Let S^ρ be a scheme satisfying (H_S),(H_C) and (H_M). Then, when ρ tends to 0, W^ρ converges to the unique viscosity solution of (3.5.5) uniformly on each compact subset of $[0,T] \times \overline{D}$.*

Proof. Let us define

$$\overline{W}(t,x,y) := \limsup_{\substack{[0,T]\times\overline{D}\ni(s,\xi,\gamma)\to(t,x,y)\\ \rho\to 0}} W^\rho(s,\xi,\gamma)$$

$$\text{and} \qquad \underline{W}(t,x,y) := \liminf_{\substack{[0,T]\times\overline{D}\ni(s,\xi,\gamma)\to(t,x,y)\\ \rho\to 0}} W^\rho(s,\xi,\gamma).$$

We start by proving that $\overline{W}$ is a viscosity sub-solution of equation (3.5.5) (the proof that $\underline{W}$ is a viscosity super-solution is analogous).

Let φ be in $C^{1,2}([0,T] \times \overline{D})$ and let $(\bar{t},\bar{x},\bar{y})$ be a local maximum point for $\overline{W} - \varphi$ on $[0,T] \times \overline{D}$. Without loss of generality we can assume that $(\bar{t},\bar{x},\bar{y})$ is a strict local maximum in the restricted ball $B_r := B((\bar{t},\bar{x},\bar{y}),r) \cap ([0,T] \times \overline{D})$ for a certain $r > 0$,

$$(\overline{W} - \varphi)(\bar{t},\bar{x},\bar{y}) = \max_{B_r}(\overline{W} - \varphi) = 0,$$

and $\varphi \geq 2\sup_\rho \|W^\rho\|_\infty$ outside the ball $B((\bar{t},\bar{x},\bar{y}),r)$.

We first assume that $\bar{t} \in (0,T)$. Then we claim that:

$$-\partial_t\varphi + H(\bar{t},\bar{x},D_x\varphi,D_x^2\varphi) \leq 0 \qquad \text{if } (\bar{x},\bar{y}) \in D, \tag{3.6.3a}$$

$$\text{or} \quad \min\left(-\partial_t\varphi + H(\bar{t},\bar{x},D_x\varphi,D_x^2\varphi), -\partial_y\varphi\right) \leq 0 \quad \text{if } (\bar{x},\bar{y}) \in \partial D. \tag{3.6.3b}$$

Following [84], there exists a sequence of mesh steps ρ_k and a sequence (t_{k-1}, x_k, y_k) in $(0,T)\times\overline{D}$ such that: $\rho_k \to 0$ and $(t_{k-1}, x_k, y_k) \to (\bar{t}, \bar{x}, \bar{y})$ as $k \to +\infty$, and (t_{k-1}, x_k, y_k) is a global maximum point of $W^{\rho_k} - \varphi$, with

$$(W^{\rho_k} - \varphi)(t_{k-1}, x_k, y_k) = \max_{[0,T]\times\overline{D}} (W^{\rho_k} - \varphi) = \delta_k \stackrel{k\to\infty}{\longrightarrow} 0 \tag{3.6.4}$$

and

$$W^{\rho_k}(t_{k-1}, x_k, y_k) \stackrel{k\to\infty}{\longrightarrow} \overline{W}(\bar{t}, \bar{x}, \bar{y}).$$

- Case 1: assume that $\bar{y} > g(\bar{x})$. Thus, $(\bar{x}, \bar{y})$ is in the open set D and for k large enough, $(x_k, y_k) \in D$. By continuity of g, $y > g(x)$ in $B((t_{k-1}, x_k, y_k), \eta(\rho_k))$ for ρ_k small enough. Therefore $\Pi^{\mathcal{G}_\rho}(W^{\rho_k}) = W^{\rho_k} \leq \varphi + \delta_k$ on $B((t_{k-1}, x_k, y_k), \eta(\rho_k))$. On the other hand, $W^{\rho_k}(t_{k-1}, x_k, y_k) = \varphi(t_{k-1}, x_k, y_k) + \delta_k$. Hence, thanks to the monotonicity of the scheme, it follows

$$\begin{aligned} 0 &= S^{\rho_k}(t_{k-1}, x_k, y_k, W^{\rho_k}(t_{k-1}, x_k, y_k), \Pi^{\mathcal{G}_\rho}(W^{\rho_k})) \\ &\geq S^{\rho_k}(t_{k-1}, x_k, y_k, \varphi(t_{k-1}, x_k, y_k) + \delta_k, \varphi + \delta_k). \end{aligned}$$

Using the consistency of the scheme, we obtain in the limit when $\rho_k \to 0$,

$$-\partial_t \varphi + H(\bar{t}, \bar{x}, D_x\varphi, D_x^2\varphi) \leq 0. \tag{3.6.5}$$

We conclude that (3.6.3a) is satisfied when $(\bar{x}, \bar{y}) \in D$.

- Case 2: when $\bar{y} = g(\bar{x})$, (x_k, y_k) can be also on ∂D and the scheme may involve values $W^n_{k\ell}$ on some points (x_k, y_ℓ) that are not in $\overline{D}$. Here, we need to consider two sub-cases.

- *Sub-Case 2.1*: if $-\partial_y \varphi(\bar{t}, \bar{x}, \bar{y}) \leq 0$, then (3.6.3b) holds.

- *Sub-Case 2.2*: if $-\partial_y \varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, then there exists a neighborhood $\mathcal{V}$ of $(\bar{t}, \bar{x}, \bar{y})$ where $\partial_y \varphi(t, x, y)$ is well defined, and $-\partial_y \varphi(t, x, y) > 0$ for every $(t, x, y) \in \mathcal{V}$. Therefore,

$$y \leq y' \Longrightarrow \varphi(t, x, y) \geq \varphi(t, x, y') \quad \forall (t, x, y), (t, x, y') \in \mathcal{V}. \tag{3.6.6}$$

For k large enough, $B((t_{k-1}, x_k, y_k), \eta(\rho_k)) \subset \mathcal{V}$.
Let $(t, x, y) \in B((t_{k-1}, x_k, y_k), \eta(\rho_k))$. If $y \geq g(x)$, then

$$\Pi^{\mathcal{G}_\rho}(W^{\rho_k}(t, x, y)) = W^{\rho_k}(t, x, y) \leq \varphi(t, x, y) + \delta_k. \tag{3.6.7}$$

Otherwise if $y < g(x)$, $\Pi^{\mathcal{G}_\rho}(W^{\rho_k}(t, x, y)) = W^{\rho_k}(t, x, y_{j_x})$, and we have

$$\begin{aligned} \Pi^{\mathcal{G}_\rho}(W^{\rho_k}(t, x, y)) &= W^{\rho_k}(t, x, y_{j_x}) \\ &\leq \varphi(t, x, y_{j_x}) + \delta_k \quad \text{(by using (3.6.4))} \qquad (3.6.8) \\ &\leq \varphi(t, x, y) + \delta_k \quad \text{(by using (3.6.6))} \qquad (3.6.9) \end{aligned}$$

(for (3.6.9) we also used the continuity of g that ensures that for k big enough (t, x, y_{j_x}) still lays in $\mathcal{V}$). We conclude $\Pi^{\mathcal{G}_\rho}(W^{\rho_k}) \leq \varphi + \delta_k$ on $B((t_{k-1}, x_k, y_k), \eta(\rho_k))$. Thus by monotonicity (H_M), we get

$$\begin{aligned} 0 &= S^{\rho_k}(t_{k-1}, x_k, y_k, W^{\rho_k}(t_{k-1}, x_k, y_k), \Pi^{\mathcal{G}_\rho}(W^{\rho_k})) \\ &\geq S^{\rho_k}(t_{k-1}, x_k, y_k, \varphi(t_{k-1}, x_k, y_k) + \delta_k, \varphi + \delta_k). \end{aligned}$$

Using the consistency of the scheme, when $\rho_k \to 0$, we get

$$-\partial_t \varphi + H(\bar{t}, \bar{x}, D_x \varphi, D_x^2 \varphi) \leq 0. \tag{3.6.10}$$

As conclusion for case 2, (3.6.3b) is satisfied when $(\bar{x}, \bar{y}) \in \partial D$.

Similar arguments can be used in order to treat the case $\bar{t} = T$ (classical arguments enable to treat also the case of the boundary $\bar{t} = 0$). We conclude that $\overline{W}$ is a viscosity sub-solution of the equation (3.5.5).

As already mentioned, we can also show that $\underline{W}$ is a super-solution. Then by the comparison principle on $[0,T] \times \overline{D}$ (Theorem 3.5.4), the inequality $\underline{W} \geq \overline{W}$ holds on $[0,T] \times \overline{D}$. Furthermore since the reverse inequality $\underline{W} \leq \overline{W}$ is always true, we deduce that $\overline{W} = \underline{W} = \vartheta$ on $[0,T] \times \overline{D}$. Hence the convergence result is proved. □

3.6.2 A semi-Lagrangian scheme

We first introduce some new notations: for any $u \in U$, the drift $b(\cdot, \cdot, u)$ (resp. volatility $\sigma(\cdot, \cdot, u)$) will be simply denoted by b^u (resp. σ^u).

Consider the operator $\Psi : C(\mathbb{R}^{d+1}) \to C([0,T] \times \mathbb{R}^{d+1})$ defined by:

$$\Psi(\phi)(t,x,y) = \min_{u \in U} \frac{1}{2p} \sum_{k=1}^{2p} [\phi]_x \Big(x + h b^u(t,x) + \sqrt{hp}(-1)^k \sigma^u_{\lfloor \frac{k+1}{2} \rfloor}(t,x), y \Big) \tag{3.6.11}$$

where $(\sigma^u_k)_{k=1,\ldots,p}$ are the column vectors of σ^u, and $\lfloor q \rfloor$ denotes the integer part of q. The notation $[\cdot] \equiv [\cdot]_x$ stands for a monotone, P_1 interpolation operator on the x-grid (x_i), satisfying for every Lipschitz continuous function ϕ (with Lipschitz constant L_ϕ):

$$\begin{cases} (i)\ [\phi]_x(x_i) = \phi(x_i),\ \forall i, \\ (ii)\ |[\phi]_x(x) - \phi(x)| \leq L_\phi |\Delta x|, \\ (iii)\ |[\phi]_x(x) - \phi(x)| \leq C|\Delta x|^2 \|D_x^2 \phi\|_\infty \ \text{ if } \phi \in C^2(\mathbb{R}^d), \\ (iv)\ \text{for any functions } \phi_1, \phi_2 : \mathbb{R}^d \to \mathbb{R},\ \phi_1 \leq \phi_2 \Rightarrow [\phi_1]_x \leq [\phi_2]_x. \end{cases} \tag{3.6.12}$$

The operator Ψ corresponds to a discretization of the equation (3.5.5a) by a semi-Lagrangian scheme (see [141, 68, 92]). Now, define an approximation method for the system (3.5.5) as follows:

Fully discrete scheme (FDS)

1) Initialization step: for all i,j, set $W^N_{i,j} = \psi(x_i, y_j \vee g(x_i))$.

2) For $n = N, \ldots, 1$:

- **Step 1** Compute $W^{n-\frac{1}{2}}_{i,j} = \Psi(W^n_{\cdot,\cdot})(t_n, x_i, y_j)$, for all i,j (here $W^n_{\cdot,\cdot}$ denotes the values $\{W^n_{i,j} \mid i,j \in \mathbb{Z}^d \times \mathbb{Z}\}$);
- **Step 2** Compute $W^{n-1}_{i,j} = \Pi^{\mathcal{G}^\rho}(W^{n-\frac{1}{2}})$, for all i,j.

The FDS scheme is a particular GS method, with the following formula for the operator S^ρ:

$$S^\rho(t-h, x, y, r, \Phi) := \frac{1}{h} \Big\{ r - \Pi^{\mathcal{G}^\rho}(\Psi(\Phi)(t,x,y)) \Big\}. \tag{3.6.13}$$

Remark 3.6.2. We point out that in the FDS, for j such that $y_j < g(x_i)$, the projection step fixes the value of $W^{n-1}_{i,j}$ as $W^{n-1}_{i,j_{x_i}}$. Of course, other alternatives could be considered, for instance, one could consider the scheme:

1) Initialization step: for $n = N$, for all i, j, set $\widetilde{W}^N_{i,j} = \psi(x_i, y_j \vee g(x_i))$.

2) For $n = N, \ldots, 1$, $\widetilde{W}^{n-1}_{i,j} = \widetilde{\Psi}(\widetilde{W}^n_{\cdot,\cdot})(t_n, x_i, y_j)$, for all i, j, where

$$\widetilde{\Psi}(\phi)(t,x,y) = \min_{u\in U} \frac{1}{2p}\sum_{k=1}^{2p}[\phi]_{x,y}\big(x+hb^u(t,x)+\sqrt{hp}(-1)^k\sigma^u_{\lfloor\frac{k+1}{2}\rfloor}(t,x), y\vee g(x)\big),$$

with $[\cdot]_{x,y}$ standing for an interpolation in both variables x and y.

All the convergence results stated in the sequel also hold for this new scheme.

In the next subsection, we will study some properties of the approximated solution W^ρ. Before this, we define also the following semi-discrete scheme where we consider only a time-discretization:

Semi-discrete scheme (SDS)

1) For $n = N$, for every $(x, y) \in \mathbb{R}^{d+1}$, set $V^N(x, y) = \psi(x, y \vee g(x))$.

2) For $n = N, \ldots, 1$, define V^{n-1} as the function defined by:

$$V^{n-1}(x,y) = \Psi_0(V^n)(t_n, x, y \vee g(x)), \tag{3.6.14}$$

where, Ψ_0 is defined from $C(\mathbb{R}^{d+1})$ into $C([0,T] \times \mathbb{R}^{d+1})$ by:

$$\Psi_0(\phi)(t,x,y) := \min_{u\in U} \frac{1}{2p}\sum_{k=1}^{2p}\phi\left(x + hb^u(t,x) + (-1)^k\sqrt{hp}\sigma^u_{\lfloor\frac{k+1}{2}\rfloor}(t,x), y\right).$$

Unlike the fully discrete scheme, no interpolation step is required in the SDS. Straightforward calculations lead to the following consistency estimate, for any $\phi \in C^{2,4}((0,T) \times \mathbb{R}^d \times \mathbb{R})$:

$$\begin{aligned} &\left|\frac{1}{h}(\phi(t-h,x,y) - \Psi_0(\phi)(t,x,y)) - \Big(-\partial_t\phi + H(t,x,D_x\phi,D^2_x\phi)\Big)\right| \\ &\leq K_1 \max_{u\in U}\Big(\|\partial^2_{tt}\phi\|_\infty + |b^u(t,x)|^2\|D^2_x\phi\|_\infty + |b^u(t,x)||\sigma^u(t,x)|^2\|D^3_x\phi\|_\infty \\ &\qquad + |\sigma^u(t,x)|^4\|D^4_x\phi\|_\infty\Big)\, h, \end{aligned} \tag{3.6.15}$$

for any $(x, y) \in \mathbb{R}^d$, with K_1 a positive constant independent of ϕ. Moreover, Ψ and Ψ_0 satisfy the following properties :

Lemma 3.6.3. *For every $\phi_1, \phi_2 \in C([0,T] \times \mathbb{R}^d \times \mathbb{R})$, we have:*

(i) $(\Psi_0(\phi_1) - \Psi_0(\phi_2))_+ \leq \|(\phi_1 - \phi_2)_+\|$;

(ii) $\|\Psi\phi_1 - \Psi\phi_2\|_\infty \leq \|\phi_1 - \phi_2\|_\infty$;

(iii) $\|\Psi_0\phi_1 - \Psi_0\phi_2\|_\infty \leq \|\phi_1 - \phi_2\|_\infty$;

(iv) for any Lipschitz continuous function ϕ on $\mathbb{R}^d \times \mathbb{R}$, $\|\Psi\phi - \Psi_0\phi\|_\infty \leq L_\phi|\Delta x|$, where L_ϕ is a Lipschitz constant of ϕ.

Notice that assertion (i) corresponds to a *discrete comparison principle.* The proof of the lemma is straightforward and will be omitted.

3.6.3 Convergence of the SL scheme

The aim of this subsection is to prove a convergence result for the scheme FDS.

Theorem 3.6.4. *Let assumptions* ($H'_{b,\sigma}$), (H'_{ψ}) *and* (H_g) *be satisfied. Let* W^ρ *be defined by the fully discrete scheme. Assume also that*

$$\frac{|\Delta x|^2}{h} \to 0.$$

Then the FDS is convergent, i.e. W^ρ *converges towards* ϑ*, as* ρ *tends to* 0*, uniformly on each compact set of* $[0,T] \times \overline{D}$.

Proof. Being ψ a bounded function, the stability of the scheme follows by $\|W^n\|_\infty \leq \|\psi\|_\infty$. Moreover, the consistency of the scheme comes from the estimates (3.6.15),(3.6.12) and by using that $\frac{|\Delta x|^2}{h} \to 0$. Also, the monotonicity property (as defined in (H_M)) is deduced from the monotonicity of the interpolations and the monotonicity of the operator Ψ_0.

In consequence, the convergence result follows from Theorem 3.6.1. □

We conclude the section with some further results on the semi-discrete scheme.

Proposition 3.6.5. *If assumptions* ($H'_{b,\sigma}$), (H'_{ψ}) *and* (H_g) *are satisfied, the solution* V *of the semi-discrete scheme is Lipschitz continuous in* (x,y) *and* $\frac{1}{2}$*-Hölder continuous in* t*: there exists a constant* $L_V \geq 0$*, for every* $0 \leq n, m \leq N$*, and every* $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^{d+1}$ *such that*

$$|V^n(x_2,y_2) - V^m(x_1,y_1)| \leq L_V\big(|x_2-x_1| + |y_2-y_1| + (1+|x_1|)|t_n-t_m|^{1/2}\big).$$

Proof. Let $(\xi^n, \xi^{n+1}, \dots, \xi^{N-1})$, be independent random variables, with values in $\{0,\dots,2p\}$, and such that $\mathbb{P}(\xi^n = k) = \frac{1}{2p}, \quad \forall k = 1,\dots,2p$.

Let $u = (u_n, u_{n+1}, \dots, u_{N-1})$ denotes a sequence of controls, with $u_i \in U$, and let $Z_k^n = Z_k^{n,x,u}$, $n \leq k \leq N$, be the random walk such that:

$$\begin{cases} Z_n^n = x, \\ Z_{k+1}^n = Z_k^n + hb^{u_k}(t_{k+1}, Z_k^n) + \sqrt{h}\bar{\sigma}_{\xi^k}^{u_k}(t_{k+1}, Z_k^n), \ k \geq n \end{cases} \tag{3.6.16}$$

where the notation $\bar{\sigma}_k^u(t,x) := (-1)^k \sqrt{p}\sigma^u_{\lfloor \frac{k+1}{2} \rfloor}(t,x)$ is used. Notice that $Z_k^{n,x,u}$ depends only on the controls $(u_n, \dots, u_{k-1})$. Direct calculations lead to the following expressions:

$$\begin{aligned} V^n(x,y) &= \min_{u_n \in U} \frac{1}{2p}\sum_{k=1}^{2p} V^{n+1}\Big(x + hb^{u_n}(t_{n+1},x) + \sqrt{h}\bar{\sigma}_k^{u_n}(t_n,x),\ y \vee g(x)\Big) \\ &= \min_{u_n \in U} \mathbb{E}\Big[V^{n+1}\big(Z_{n+1}^{n,x,u},\ y \vee g(x)\big)\Big], \end{aligned}$$

and, in the same way,

$$\begin{aligned} V^n(x,y) &= \min_{u_n \in U} \mathbb{E}\Big[\min_{u_{n+1} \in U} \mathbb{E}\big[V^{n+2}(Z_{n+2}^{n+1, Z_{n+1}^{n,x,u}, u}, y \vee g(x) \vee g(Z_{n+1}^{n,x,u}))\big]\Big], \\ &= \min_{u_n \in U} \mathbb{E}\Big[\min_{u_{n+1} \in U} \mathbb{E}\big[V^{n+2}(Z_{n+2}^{n,x,u}, y \vee g(x) \vee g(Z_{n+1}^{n,x,u}))\big]\Big], \end{aligned}$$

and so on, and finally (also using $V^N(x,y) \equiv \psi(x, y \vee g(x))$):

$$\begin{aligned} V^n(x,y) =& \min_{u_n} \mathbb{E}\Big[\min_{u_{n+1}} \mathbb{E}\Big[\cdots \min_{u_{N-1}} \mathbb{E}\Big[V^N\big(Z_N^{n,x,u}, y \vee \max_{i=n,\dots,N-1} g(Z_i^{n,x,u})\big)\Big]\Big]\cdots\Big] \\ =& \min_{u_n} \mathbb{E}\Big[\min_{u_{n+1}} \mathbb{E}\Big[\cdots \min_{u_{N-1}} \mathbb{E}\Big[\psi\big(Z_N^{n,x,u}, y \vee \max_{i=n,\dots,N} g(Z_i^{n,x,u})\big)\Big]\Big]\cdots\Big]. \end{aligned}$$

Then, we have the equivalent representation formula:

$$V^n(x,y) = \min_{u=(u_n,\dots,u_{N-1})} \mathbb{E}\Big[\psi(Z_N^{n,x,u}, y \vee \max_{i=n,\dots,N} g(Z_i^{n,x,u}))\Big]. \tag{3.6.17}$$

By using $|(x_2 \vee y_2) - (x_1 \vee y_1)| \le |x_1 - x_2| \vee |y_1 - y_2| \le |x_1 - x_2| + |y_1 - y_2|$, we obtain:

$$\begin{aligned} &|V^n(x_2,y_2) - V^n(x_1,y_1)| \\ \le& L_\psi \max_u \mathbb{E}\Big[|Z_N^{n,x_2,u} - Z_N^{n,x_1,u}| + L_g \max_{n\le i\le N} |Z_i^{n,x_2,u} - Z_i^{n,x_1,u}| + L_g|y_2 - y_1|\Big] \\ \le& L_\psi(L_g+1)\Big(\max_u \mathbb{E}\Big[\max_{n\le i\le N} |Z_i^{n,x_2,u} - Z_i^{n,x_1,u}|\Big] + |y_2 - y_1|\Big), \end{aligned} \tag{3.6.18}$$

where $\max_u$ denotes the maximum over $(u_n, u_{n+1}, \dots, u_{N-1}) \in U^{N-n}$. By ($H'_{b,\sigma}$), there exists a constant $C_1 \ge 0$, that depends only on (T, M_σ, M_b), such that for every $0 \le m, n \le N$ and every $x, x_1, x_2 \in \mathbb{R}^d$, the following classical estimates hold:

$$\mathbb{E}\Big[\max_{n\le i\le N} \big|Z_i^{n,x_2,u} - Z_i^{n,x_1,u}\big|\Big] \le C_1|x_2 - x_1| \tag{3.6.19a}$$

$$\mathbb{E}\Big[\max_{0\le i\le p} \big|Z_{m+i}^{n,x,u} - Z_m^{n,x,u}\big|\Big] \le C_1|t_{m+p} - t_m|^{1/2}(1+|x|). \tag{3.6.19b}$$

Combining (3.6.18) and (3.6.19a), for $n = m$, the Lipschitz property follows:

$$|V^n(x_1,y_1) - V^n(x_2,y_2)| \le L_\psi(L_g+1)(C_1|x_2 - x_1| + |y_2 - y_1|).$$

On the other hand, for $0 \le n \le m \le N$ by using again (3.6.17) we have:

$$|V^m(x,y) - V^n(x,y)| \le C \max_u \mathbb{E}\Big[\Big|\max_{n\le i\le N} Z_i^{n,x,u} - \max_{m\le i\le N} Z_i^{m,x,u}\Big|\Big]. \tag{3.6.20}$$

for some constant C. Therefore, since $Z_m^{m,x,u} = x$:

$$|V^m(x,y) - V^n(x,y)| \le C \max_u \mathbb{E}\Big[\Big|\max_{n\le i\le m} Z_i^{n,x,u} - x\Big|\Big] \tag{3.6.21}$$

$$+ C \max_u \mathbb{E}\Big[\Big|\max_{m+1\le i\le N} Z_i^{n,x,u} - \max_{m+1\le i\le N} Z_i^{m,x,u}\Big|\Big]. \tag{3.6.22}$$

The right term of (3.6.21) is bounded by $C(1+|x|)|t_m - t_n|^{1/2}$. In order to bound (3.6.22), by using that $Z_i^{n,x,u} = Z_i^{m,Z_m^{n,x,u},u}$ ($\forall i \ge m$) and (3.6.19a) the following estimate is obtained:

$$\begin{aligned} \mathbb{E}\Big[\Big|\max_{m+1\le i\le N} Z_i^{m,Z_m^{n,x,u},u} - \max_{m+1\le i\le N} Z_i^{m,x,u}\Big|\Big] &\le C_1 \mathbb{E}\big[Z_m^{n,x,u} - x\big] \\ &\le C_1^2(1+|x|)|t_m - t_n|^{1/2}. \end{aligned}$$

Hence it holds, for some constant C,

$$|V^n(x,y) - V^m(x,y)| \leq C(1+|x|)|t_m - t_n|^{1/2}.$$

Together with the Lipschitz property the desired result follows. □

Remark 3.6.6. It is not clear whether the solution W^ρ obtained by the fully discrete scheme satisfies a Lipschitz continuity property or not. The main difficulty is that a representation formula as (3.6.17) is not guaranteed for W^ρ.

Proposition 3.6.7. *If assumptions* ($H'_{b,\sigma}$),(H'_ψ) *and* (H_g) *are satisfied, there exists a constant* $C>0$ *independent of* ρ *such that,* $\forall n = 0,\dots,N$*:*

$$\|W^n - V^n\|_\infty \leq C\frac{|(\Delta x, \Delta y)|}{h}. \tag{3.6.23}$$

Proof. Consider the operator $\Pi : C(\mathbb{R}^d\times\mathbb{R}) \to C(\mathbb{R}^d\times\mathbb{R})$ defined by $\Pi\phi(x,y) = \phi(x, y\vee g(x))$. With this notation, we have $V^{n-1} = \Pi(\Psi_0(V^n))$. On the other hand, $W^{n-1}_{ij} = \Pi^{\mathcal{G}^\rho}(\Psi(W^n)(t_n,x_i,y_j))$. Therefore, by using Lemma 3.6.3, we get:

$$\begin{aligned}
\|W^{n-1} - V^{n-1}\|_\infty &\leq \|\Pi^{\mathcal{G}^\rho}(\Psi(W^n)) - \Pi(\Psi_0(V^n))\|_\infty \\
&\leq \|\Pi^{\mathcal{G}^\rho}(\Psi(W^n)) - \Pi^{\mathcal{G}^\rho}(\Psi_0(V^n))\|_\infty + \|\Pi^{\mathcal{G}^\rho}(\Psi_0(V^n)) - \Pi(\Psi_0(V^n))\|_\infty \\
&\leq \|\Psi(W^n) - \Psi_0(V^n)\|_\infty + \|\Pi^{\mathcal{G}^\rho}(\Psi_0(V^n)) - \Pi(\Psi_0(V^n))\|_\infty \\
&\leq \|\Psi(W^n) - \Psi(V^n)\|_\infty + \|\Psi(V^n) - \Psi_0(V^n)\|_\infty + L_V\Delta y \\
&\leq \|W^n - V^n\|_\infty + L_V|\Delta x| + L_V\Delta y.
\end{aligned}$$

By recursion, it follows:

$$\|W^n - V^n\|_\infty \leq \|W^N - V^N\|_\infty + nL_V(|\Delta x| + \Delta y).$$

Since $V^N(x,y) = \psi(x, y\vee g(x))$ on $\mathbb{R}^{d+1}$, W^N is an interpolation of V^N on the grid $\mathcal{G}^\rho$ and $n \leq \frac{T}{h}$, we obtain the desired result. □

3.7 Error bounds for the semi-Lagrangian scheme

In this section we aim to give an error bound for the fully discrete scheme FDS, as well as the semi-discrete SL scheme (3.6.14).

The error estimates theory is based on some technique of "shaking coefficients" and regularization introduced by Krylov in [119, 120] and studied later by many authors [31, 32, 33, 48, 49, 92]. The main idea consists of regularizing the exact solution ϑ in order to obtain a smooth sub-solution ϑ_ε (for an approximation parameter $\varepsilon > 0$) to the equation (3.5.5), then the upper-bound error estimate can be obtained by using the consistency estimate (3.6.15). The regularization procedure takes advantage from the fact that the exact solution ϑ is Hölder continuous, which enables to obtain an estimate of $|\vartheta - \vartheta_\varepsilon|_\infty$ of order $O(\varepsilon)$. The lower bound is obtained by similar arguments, but in this case we need to build a smooth sub-solution to the discrete equation. For this, a regularization of the numerical solution will be considered. However, as noticed earlier (Remark 3.6.6), it is not clear if the solution W^ρ of the fully discrete scheme is Hölder continuous or not, and then it is not clear how to build directly a regularization W^ρ_ε (with regularization parameter ε) with an error $|W^\rho - W^\rho_\varepsilon|$ of order $O(\varepsilon)$. For this reason, we will start by deriving an error estimate between the solution ϑ and the solution V of the semi-discrete scheme SDS, using the fact that V has Hölder continuity properties. Then the estimates for the FDS are derived as a consequence of Lemma 3.6.7.

For the regularization procedure, consider a smooth function $\mu : \mathbb{R}^{d+2} \to \mathbb{R}$, with $\mu \geq 0$, supported in $(0,1) \times B_1(0)$, with $\int_{\mathbb{R}} \int_{\mathbb{R}^d} \mu(s,x)\, dxds = 1$, and define μ_ε as the following sequence of mollifiers:

$$\mu_\varepsilon(t,x,y) := \frac{1}{\varepsilon^{d+3}} \mu(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}) \qquad \text{in } \mathbb{R} \times \mathbb{R}^{d+1}.$$

3.7.1 The rate of convergence for the semi-discrete scheme

For any $\varepsilon > 0$, let $\mathcal{E}$ be the set of progressively measurable processes (α, χ) valued in $[-\varepsilon^2, 0] \times B(0,\varepsilon) \subset \mathbb{R} \times \mathbb{R}^d$ that is,

$$\mathcal{E} := \{\text{prog. meas. process } (\alpha,\chi) \text{ valued in } E\},$$

where $E := \left\{(a,e) \in \mathbb{R} \times \mathbb{R}^d,\ -\varepsilon^2 \leq a \leq 0, |e| \leq \varepsilon\right\}$.

On other hand, let $\overline{M} := 2\sqrt{1 + L_g^2}$ and introduce $g_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ defined by

$$g_\varepsilon(x) := g(x) - \overline{M}\,\varepsilon. \tag{3.7.1}$$

Finally, let us denote by D^ε the set defined as follows:

$$D^\varepsilon := \left\{(x,y) \in \mathbb{R}^{d+1},\ y > g_\varepsilon(x)\right\}.$$

Remark 3.7.1. The choice of g_ε is such that the following property holds:

$$(x,y) \in \overline{D} \Longrightarrow (x - e_1, y - e_2) \in D^\varepsilon \quad \forall\ (e_1,e_2) \in \mathbb{R}^{d+1}, |(e_1,e_2)| \leq \varepsilon.$$

Upper bound

Now, we start by introducing a perturbed control problem (with "shaking coefficients"). For any $\varepsilon > 0$, consider the following value function

$$w^\varepsilon(t,x,y) := \inf_{\substack{u\in\mathcal{U},\\ (\alpha,\chi)\in\mathcal{E}}} \mathbb{E}\left[\psi\left(X^{u,(\alpha,\chi)}_{t,x}(T), \max_{s\in[t,T]} g_\varepsilon(X^{u,(\alpha,\chi)}_{t,x}(s)) \vee y\right)\right], \tag{3.7.2}$$

where $X^{u,(\alpha,\chi)}_{t,x}(\cdot)$ is the solution of the perturbed system of SDEs

$$\begin{cases} dX(s) = b(s+\alpha(s), X(s)+\chi(s), u(s))ds + \sigma(s+\alpha(s), X(s)+\chi(s), u(s))d\mathcal{B}(s) \\ X(t) = x. \end{cases} \tag{3.7.3}$$

Remark 3.7.2. The functions σ and b are only defined for times $t \in [0,T]$, but they can be extended to times $[-2\varepsilon^2, T + 2\varepsilon^2]$ in such a way that assumption $(H'_{b,\sigma})$ still holds.

Proposition 3.7.3. *Under assumptions* $(H'_{b,\sigma})$,(H'_ψ) *and* (H_g) *(extending eventually* b *and* σ *as prescribed in Remark 3.7.2), the following holds:*

(i) w^ε *is a Lipschitz continuous function in* x *and* y *and a* $\frac{1}{2}$*-Hölder continuous function in* t*. More precisely, for* $t, \leq t' \in [0,T]$*,* $x, x' \in \mathbb{R}^d$ *and* $y, y' \in \mathbb{R}$*, we have:*

$$|w^\varepsilon(t,x,y) - w^\varepsilon(t',x',y')| \leq L_\vartheta(|x-x'| + |y-y'| + (1+|x|)|t-t'|^{1/2}).$$

(ii) $|\vartheta(t,x,y) - w^\varepsilon(t,x,y)| \leq C\varepsilon$ *on* $[0,T]\times\overline{D}$*, where the constant* C *only depends on* T *and the Lipschitz constants of* b*,* σ*,* g *and* ψ*.*

Proof. The Lipschitz and Hölder continuity follows by the same arguments as in Proposition 3.4.1. Thanks to the Lipschitz continuity of ψ and g and the choice of g_ε one has

$$|\vartheta(t,x,y) - w^\varepsilon(t,x,y)| \leq K\left(\sup_{\substack{u\in\mathcal{U},\\(\alpha,\chi)\in\mathcal{E}}} \mathbb{E}\left[\sup_{s\in[t,T]}\left|X^u_{t,x}(s) - X^{u,(\alpha,\chi)}_{t,x}(s)\right|\right] + \overline{M}\,\varepsilon\right)$$

and the result is obtained by classical estimates thanks to the hypothesis on b and σ. □

Theorem 3.7.4. *Let assumptions* ($H'_{b,\sigma}$),(H'_ψ) *and* (H_g) *be satisfied. There exists a constant* $C \geq 0$ *such that*

$$\vartheta(t_n,x,y) - V^n(x,y) \leq C\ h^{1/4},$$

for all $n \geq 0$ *(with* $nh \leq T$*) and* $(x,y)\in\overline{D}$*.*

Proof. We split the proof into three steps:

- *Step 1.* Let $(\tilde a,(\tilde e_1,\tilde e_2)) \in \mathbb{R}\times\mathbb{R}^{d+1}$ be such that $-\varepsilon^2 \leq \tilde a \leq 0$ and $|(\tilde e_1,\tilde e_2)| \leq \varepsilon$. Let $(\bar t,\bar x,\bar y) \in (0,T)\times D$, and define for $\nu > 0$:

$$I_\nu(\bar t,\bar x,\bar y) := \left\{(t,x,y)|t\in[\bar t-\nu^2,\bar t],(x,y)\in B_\nu(\bar x,\bar y)\right\}.$$

We claim that $w^\varepsilon(\cdot-\tilde a,\cdot-\tilde e_1,\cdot-\tilde e_2)$ is a viscosity sub-solution of

$$-\partial_t\vartheta + H(t,x,D_x\vartheta,D^2_x\vartheta) \leq 0 \quad \text{on } I_\varepsilon(\bar t,\bar x,\bar y). \tag{3.7.4}$$

To prove this claim, we notice first that $I_\varepsilon(\bar t,\bar x,\bar y)\subset D^\varepsilon$. Moreover, from Theorem 3.5.2, for any $\varepsilon > 0$, w^ε is a viscosity solution of the following equation

$$\begin{cases} -\partial_t w^\varepsilon + \sup\limits_{(a,e)\in E} H(t+a,x+e,D_x w^\varepsilon, D^2_x w^\varepsilon) = 0 & (-2\varepsilon^2,T)\times D^\varepsilon, \\ -\partial_y w^\varepsilon = 0 & (-2\varepsilon^2,T)\times\partial D^\varepsilon, \\ w^\varepsilon(T,x,y) = \psi(x,(g(x)-\overline{M}\varepsilon)\vee y). \end{cases} \tag{3.7.5}$$

Let $\varphi \in C^{2,4}([-2\varepsilon^2,T]\times\overline{D^\varepsilon})$ be such that $w^\varepsilon(\cdot-\tilde a,\cdot-\tilde e_1,\cdot-\tilde e_2)-\varphi$ achieves a local maximum at $(\tilde t,\tilde x,\tilde y)$ on $I_\varepsilon(\bar t,\bar x,\bar y)$. Clearly $(\tilde t-\tilde a,\tilde x-\tilde e_1,\tilde y-\tilde e_2)$ is also a local maximum of $w^\varepsilon-\varphi(\cdot+\tilde a,\cdot+\tilde e_1,\cdot+\tilde e_2)$ on $I_{2\varepsilon}$ (and $I_{2\varepsilon}\subset D^\varepsilon$). Since w^ε is a viscosity solution of equation (3.7.5), we obtain:

$$-\partial_t\varphi(\tilde t,\tilde x,\tilde y) + \sup_{(a,e_1)\in E} H(\tilde t-\tilde a+a,\tilde x-\tilde e_1+e_1,D_x\varphi,D^2_x\varphi) \leq 0.$$

Taking $(a,e_1) = (\tilde a,\tilde e_1)$, we get the result.

- *Step 2.* Define following mollification w_ε:

$$w_\varepsilon(t,x,y) := (w^\varepsilon * \mu_\varepsilon)(t,x,y) = \int_{\substack{|(e_1,e_2)|\le\varepsilon \\ -\varepsilon^2\le a\le 0}} w^\varepsilon(t-a, x-e_1, y-e_2)\mu(a,e_1,e_2)dade.$$

We recall the following properties of the mollifiers:

$$|w^\varepsilon(t,x,y) - w_\varepsilon(t,x,y)| \le [w^\varepsilon]_1\varepsilon, \tag{3.7.6}$$

with $[w^\varepsilon]_1 \le CL_\vartheta$, $C \ge 0$. Moreover, for any $i \ge 1$ or $j \ge 1$,

$$\|D_x^i w_\varepsilon\|_\infty \le CL_\vartheta\,\varepsilon^{1-i}, \quad \|D_t^j w_\varepsilon\|_\infty \le CL_\vartheta\,\varepsilon^{1-2j} \tag{3.7.7}$$

(where D_x^i denotes the i-th derivative with respect to x, and D_t^j the j-th derivative with respect to t). Since w_ε is a limit of convex combinations of $w^\varepsilon(\cdot-\tilde a, \cdot-\tilde e_1, \cdot-\tilde e_2)$, then w_ε satisfies in the viscosity sense

$$-\partial_t w_\varepsilon + H(t,x,D_x w_\varepsilon, D_x^2 w_\varepsilon) \le 0 \qquad \text{in } (0,T)\times D. \tag{3.7.8}$$

Taking into account that w_ε is in $C^\infty([0,T]\times\mathbb{R}^{d+1})$, we conclude that (3.7.8) holds in classical sense on $[0,T]\times\overline{D}$. From the consistency estimate (3.6.15) along with (3.7.8) and by assumption ($H'_{b,\sigma}$), we get:

$$w_\varepsilon(t_{n-1},x,y) - \Psi_0(w_\varepsilon)(t_n,x,y) \le \; C\frac{h^2}{\varepsilon^3}.$$

Combining these bounds with Lemma 3.6.3, we get

$$\begin{aligned} w_\varepsilon(t_{n-1},x,y) - V^{n-1}(x,y) &\le \Psi_0(w_\varepsilon)(t_n,x,y) - \Psi_0(V^n)(x,y) + C\frac{h^2}{\varepsilon^3} \\ &\le \|\big(w_\varepsilon(t_n,\cdot) - V^n\big)_+\|_\infty + C\frac{h^2}{\varepsilon^3}. \end{aligned}$$

Therefore, by recursion, it comes

$$\|(w_\varepsilon(t_n,\cdot) - V^n)_+\|_\infty \le \|(w_\varepsilon(0,\cdot) - V^N))_+\|_\infty + CT\frac{h}{\varepsilon^3}. \tag{3.7.9}$$

- *Step 3.* By Proposition 3.7.3 together with the inequalities (3.7.6) and (3.7.9), we obtain for $n \ge 0$,

$$\|(\vartheta(t_n,\cdot) - V^n)_+\|_\infty \le C\varepsilon + CT\frac{h}{\varepsilon^3}, \qquad n \ge 0.$$

The choice $\varepsilon^4 = h$ leads to (for $n \ge 0$):

$$\|(\vartheta(t_n,\cdot) - V^n)_+\|_\infty \le C\; h^{1/4},$$

which concludes the proof. □

Lower bound

For obtaining the lower bound we will apply exactly the same techniques as used for the upper bound, reversing the role of the equation and the scheme. The key point is that the solution V of the semi-discrete scheme SDS is Lipschitz continuous. Then it is possible to use the techniques of "shaking coefficients" and regularization to build a smooth sub-solution V_ε satisfying $\|V_\varepsilon^n - V^n\|_\infty \leq C\varepsilon$ and

$$V_\varepsilon^{n-1}(x,y) - \Psi_0(V_\varepsilon^n)(t_n, x, y \vee g(x)) \leq 0 \qquad \text{in } [0,T] \times \mathbb{R}^d \times \mathbb{R}.$$

Then by consistency estimate and comparison principle, we conclude the following result:

Theorem 3.7.5. *Let assumptions* $(H'_{b,\sigma})$,(H'_ψ) *and* (H_g) *be satisfied. There exists a constant* $C > 0$ *such that*

$$\vartheta(t_n, x, y) - V^n(x,y) \geq -C\ h^{1/4}$$

for all $n \geq 0$ *(with* $Nh \leq T$*) and* $(x,y) \in \overline{D}$.

3.7.2 The fully discrete scheme

The section is concluded with the following theorem that provides error estimates for the fully discrete semi-Lagrangian scheme. The result is a simple consequence of Theorems 3.7.4, 3.7.5 and Lemma 3.6.7.

Theorem 3.7.6. *Let assumptions* $(H'_{b,\sigma})$,(H'_ψ) *and* (H_g) *be satisfied. If* W *is the solution of the fully-discrete scheme* (3.6.13) *, there exists a constant* $C > 0$ *independent from* $n = 0, \ldots, N$ *such that*

$$\|W^n - \vartheta(t_n, \cdot)\|_\infty \leq C\left(h^{1/4} + \frac{|(\Delta x, \Delta y)|}{h}\right).$$

Proof. The result follows by using

$$\|W^n - \vartheta(t_n, \cdot)\|_\infty \leq \|W^n - V^n\|_\infty + \|V^n - \vartheta(t_n, \cdot)\|_\infty.$$

and thanks to Theorems 3.7.4, 3.7.5 and to Lemma 3.6.7. □

3.8 Numerical tests

In this section we present some numerical results for reachability problems in presence of state constraints (as in Sections 3.3 and 3.2). The dynamics is given by the following controlled SDE in $\mathbb{R}^2$:

$$\begin{cases} dX(s) = u(s)\begin{pmatrix} 1 \\ 0 \end{pmatrix} ds + u(s)\sigma(X(s))d\mathcal{B}(s), \ s \geq t, \\ X(t) = x \end{cases} \tag{3.8.1}$$

where $\mathcal{B}$ is a one-dimensional Brownian motion ($p = 1$), $U = [0,1] \subset \mathbb{R}$ and the function $\sigma(x) \in \mathbb{R}^2$ will vary depending on the example. The target set is

$$\mathcal{T} = \left\{x \equiv (x_1, x_2) \in \mathbb{R}^2,\ 0 \leq x_1 \leq 0.4,\ |x_2| \leq 0.5\right\}$$

and the set of state constraints is

$$\mathcal{K} = \mathbb{R}^2 \setminus \{x \equiv (x_1, x_2) \in \mathbb{R}^2,\ -0.4 < x_1 < -0.2,\ |x_2| < 0.1\}.$$

Given a final time T, the aim is to approximate the set $\mathcal{R}_t^{\mathcal{T},\mathcal{K}}$ defined by (3.3.2), for $t = 0$. Starting by the level set function $w_{\mathcal{T},\mathcal{K}}$ the following auxiliary value function is introduced:

$$\vartheta_{\mathcal{T},\mathcal{K}}(t, x, y) = \inf_{u \in \mathcal{U}} \mathbb{E}\Big[d_{\mathcal{T}}^{+}(X_{t,x}^{u}(T)) \vee \max_{s \in [t,T]} d_{\mathcal{K}}^{+}(X_{t,x}^{u}(s)) \vee y \Big].$$

The characterization of the backward reachable set is given by

$$x \in \mathcal{R}_t^{\mathcal{T},\mathcal{K}} \Leftrightarrow \vartheta_{\mathcal{T},\mathcal{K}}(t, x, 0) = 0. \tag{3.8.2}$$

In all the following tests, $t = 0$, $T = 0.5$ are fixed and the computational domain for (x_1, x_2, y) is

$$(x_1, x_2, y) \in \Omega = [-1, 1] \times [-1, 1] \times [0, 1].$$

The numerical scheme implemented is the semi-Lagrangian scheme (FDS). We denote by N_t the number of time-steps, $N_{x_1} = N_{x_2}$ and N_y are the number of mesh steps for the space variables (x_1, x_2, y), and

$$h := \frac{T}{N_t}, \quad \Delta x_i = \frac{2}{N_{x_i}}, \quad \Delta y = \frac{1}{N_y}.$$

Different simulations show that the results are not very sensitive to the step discretization of the variable y. Unless otherwise precised, we set $N_y = 10$. For all tests, since $u \in [0, 1]$ and because the dynamics depends linearly on the control u it is sufficient to take only two controls ($N_u = 2$), e.g., $u \in \{0, 1\}$ for the discretization of the control variable.

The different figures (see e.g. Figure 3.1) represent points in the (x_1, x_2) plane. The obstacle is represented in black, and the target in dark grey. Then, an arbitrary non negative threshold $\epsilon := 10^{-5}$ is choosen, and our approximation of the reachable set is given by

$$\mathcal{R}_t^{\mathcal{T},\mathcal{K}}(\epsilon) := \Big\{ (x_1, x_2) \in \mathbb{R}^2,\ \vartheta_{\mathcal{T},\mathcal{K}}(0, (x_1, x_2), \epsilon) \leq \epsilon \Big\} \tag{3.8.3}$$

(plotted in green).

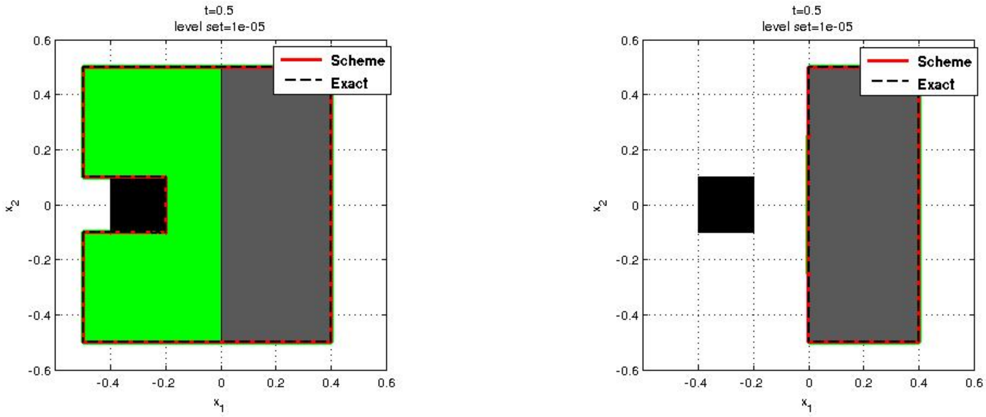
Remark 3.8.1. This procedure is not very sensitive to the choice of ϵ in the range 10^{-2} – 10^{-6}.

Remark 3.8.2. We have also tested the scheme of Remark 3.6.2 and similar results have been obtained.

Test 1. In this test, there is no diffusion, that is:

$$\sigma(x) := \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Figure 3.1(a) shows the approximation obtained using $(N_{x_1}, N_{x_2}, N_t) = (800, 800, 200)$ (and the represented level set is 10^{-5}).



(a) (b)

Figure 3.1: (a): Test 1, no diffusion, and (b): Test 2, with diffusion.

The boundary of $\mathcal{R}_t^{\mathcal{T},\mathcal{K}}$ (resp. $\mathcal{R}_t^{\mathcal{T},\mathcal{K}}(\epsilon)$) is also represented by a black doted line (resp. red line). The result perfectly matches the expected solution.

Test 2. In this test a non-zero volatility is considered:

$$\sigma(x) := \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

In that case the backward reachable set reduces to the target set, see Figure 3.1(b). In fact for any point outside the target, as soon as $u \neq 0$, even if the drift steers the system in the direction of the target, there is always a non-zero probability to go to far way in the x_2 orthogonal direction and therefore to not reach it.

Test 3. In this test the volatility is now given by

$$\sigma(x) = 5\, d_\Theta^+(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where d_Θ^+ denotes the positive distance function to the set

$$\Theta := \big\{(x_1, x_2),\ |x_2| \geq 0.3\big\}.$$

Hence for any point (x_1, x_2), if $|x_2| \geq 0.3$ the volatility vanishes. According to the drift term, the target is finally reached, see Figure 3.2. This Figure shows the approximation of $\widetilde{\mathcal{R}}_t^{\mathcal{T},\mathcal{K}}$ for three different meshes. It also shows how the scheme converges. Notice that the points which are immediately above or below the obstacle are not in the reachable set since in presence of diffusion the state constraint will be violated with a non-zero probability.

Also, in Table 3.1, various error norms are given to study the convergence of the scheme. For a given $n \geq 1$ we have chosen

$$N_{x_1} = N_{x_2} = n, \qquad N_t = n/4 \qquad \text{and} \qquad N_y = n/4.$$

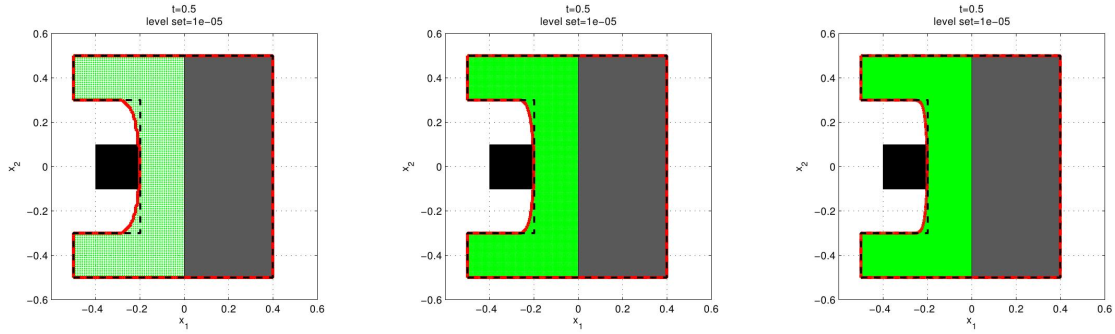


Figure 3.2: (Test 3) Vertical diffusion, using $(N_{x_1}, N_{x_2}, N_t) = (n, n, n/4)$, for $n \in \{200, 400, 800\}$.

Table 3.1: Test 3, convergence table.

n	L^∞-error	order	L^1-error	order	L^2-error	order
10	0.46582	-	0.02526	-	0.07115	-
20	0.16633	1.48	0.00345	2.87	0.01979	1.84
40	0.06746	1.30	0.00111	1.63	0.00668	1.56
80	0.02500	1.43	0.00024	2.20	0.00194	1.78

Errors have been computed by taking $n = 160$ for the reference value, and a convergence of order greater than one is observed in this simple example (better than the expected order of convergence).

Test 4 (oblique diffusion) In this test the coefficient σ is now given by

$$\sigma(x) = 5\, d_\Theta^+(x) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.8.4}$$

In Figure 3.3 we have plotted the results obtained with three different meshes, using $(N_{x_1}, N_{x_2}, N_t) = (n, n, n/4)$ for $n \in \{100, 200, 400\}$. Although the first computation plotted in Figure 3.3 (left, with $n = 100$) is not very accurate, the other computations (with $n = 200$ and $n = 400$) clearly show the good convergence of the scheme.

3.9 Appendix: A result of existence of optimal controls for linear stochastic differential equations

At the basis of the approach presented in Section 3.2 there is the equivalence

$$x \in \mathcal{R}_t^{\mathcal{T},\mathcal{K}} \Leftrightarrow v_{\mathcal{T},\mathcal{K}}(t,x) \le 0, \tag{3.9.1}$$

that permits to characterize $\mathcal{R}_t^{\mathcal{T},\mathcal{K}}$ as the 0-level set of the value function

$$v_{\mathcal{T},\mathcal{K}}(t,x) = \inf_{u \in \mathcal{U}} \mathbb{E}\Big[g_{\mathcal{T}}(X_{t,x}^u(T)) \vee \max_{s\in[t,T]} g_{\mathcal{K}}(X_{t,x}^u(s)) \Big]. \tag{3.9.2}$$

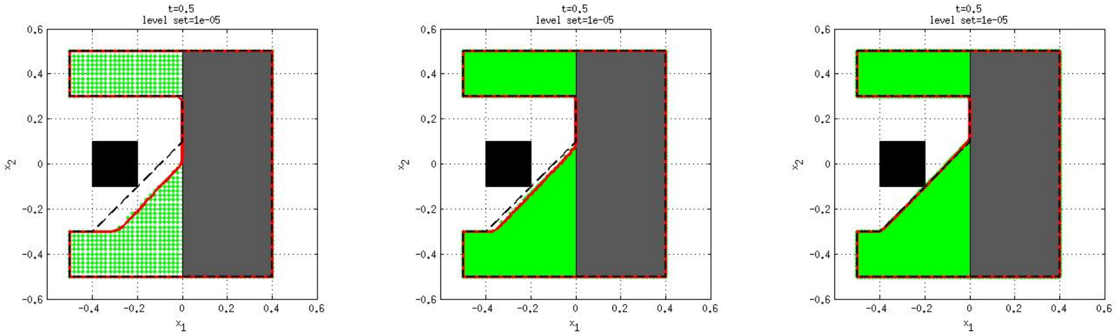


Figure 3.3: (Test 4) Oblique diffusion. using $(N_{x_1}, N_{x_2}, N_t) = (n, n, n/4)$ for $n \in \{100, 200, 400\}$.

It was already pointed out in Section 3.3 the necessity of the existence of an optimal control for (3.9.2) in order to prove this equivalence. For this reason we aim here to provide an existence result for optimal control problem with a cost of the form

$$J(t,x,u) := \mathbb{E}\left[\psi(X^u_{t,x}(T)) \vee \max_{s\in[t,T]} g(X^u_{t,x}(s))\right]. \tag{3.9.3}$$

in the case of systems governed by linear equations. The proof of the result is a simple adaptation to the cost functional (3.9.3) of the arguments given in [172, Theorem 5.2, Chapter II] that we report below for completeness. We will work in the following setting:

$$(E_1) \quad \begin{cases} (i) & \mathcal{B} \text{ is a one-dimensional Brownian motion, that is } p = 1; \\ (ii) & b, \sigma : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d \text{ are given by:} \\ & \quad b(t,x,u) = A(t)x + B(t)u, \\ & \quad \sigma(t,x,u) = C(t)x + D(t)u \\ & \text{where } A, B, C \text{ and } D \text{ are } L^\infty\text{continuous functions} \\ & \text{with value in matrix spaces of suitable sizes.} \end{cases}$$

Let us also consider the following convexity assumptions:

$$(E_2) \qquad U \subset \mathbb{R}^m \text{ is a convex and compact set;}$$

$$(E_3) \qquad g_{\mathcal{T}} \text{ and } g_{\mathcal{K}} \text{ are Lipscitz and convex functions.}$$

Remark 3.9.1. In the study of the reachability analysis, a typical choice for the functions ψ and g is $\psi = d^+_{\mathcal{T}}$ and $g = d^+_{\mathcal{K}}$ (Remark 3.3.1). In this case, if $\mathcal{T}$ and $\mathcal{K}$ are nonempty closed and convex sets, assumption (E_3) is automatically satisfied.

Theorem 3.9.2 (Existence of optimal control)**.** *Consider assumptions (E_1)-(E_3). Then for any $t \in [0,T], x \in \mathbb{R}^d$ such that*

$$v(t,x) = \inf_{u\in\mathcal{U}} J(t,x,u)$$

is finite, there exists an optimal control.

Proof. Let us consider a minimizing sequence of controls $(u_j)_{j\geq 1}$ such that

$$v(t,x) = \lim_{j\to\infty} J(t,x,u_j).$$

Thanks to the compactness of the set U, there exists a constant $C > 0$ such that

$$\mathbb{E}[\int_t^T |u_j(s)|^2 ds] \leq C$$

so

$$u_j(\cdot) \rightharpoonup \bar{u}(\cdot) \qquad \text{weakly in } \mathbb{L}^2_{\mathcal{F}}\text{-norm.}$$

From Mazur's lemma, there exists a convex combination

$$\tilde{u}_j = \sum_{i\geq 1} \lambda_{ij} u_{i+j}, \qquad \lambda_{ij} \geq 0, \ \sum_{i\geq 1} \lambda_{ij} = 1,$$

such that

$$\tilde{u}_j(\cdot) \longrightarrow \bar{u}(\cdot) \qquad \text{strongly in } \mathbb{L}^2_{\mathcal{F}}\text{-norm.}$$

Thanks to convexity and closure of U we have that $\tilde{u}_j(s)$ and $\bar{u}(s)$ still belong to U for every $s \in [t,T]$.
From the linearity of our problem we can also state that

$$X^{\tilde{u}_j}_{t,x}(\cdot) = \sum_{i\geq 1} \lambda_{ij} X^{u_{i+j}}_{t,x}(\cdot)$$

Moreover with standard passages we can show that if $X^u_{t,x}(\cdot)$ and $X^\nu_{t,x}(\cdot)$ are the strong solutions respectively associated to the controls u and ν then there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{s\in[t,T]} |X^u_{t,x}(s) - X^\nu_{t,x}(s)|^2\right] \leq C \|u-\nu\|_{\mathbb{L}^2_{\mathcal{F}}}. \tag{3.9.4}$$

Then

$$X^{\tilde{u}_j}_{t,x}(\cdot) \longrightarrow X^{\bar{u}}_{t,x}(\cdot) \qquad \text{strongly in } C([0,T],\mathbb{R}^d).$$

In what follows we will make use of the following two properties:

(*i*) $\max\left(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right) \leq \lim_{n\to\infty} \max(a_n,b_n)$;

(*ii*) $\max\left(\sum_n \lambda_n x_n, \sum_n \lambda_n y_n\right) \leq \sum_n \lambda_n \max(x_n,y_n)$, for any $\lambda_n \geq 0, \forall n \in \mathbb{N}$.

Recalling that by assumption ψ and g are convex functions we have that for any $\varepsilon > 0$ there is $\bar{j} = \bar{j}(\varepsilon)$ such that for any $j > \bar{j}$

$$\begin{aligned}
&J(t,x,\bar{u}) \\
&= \mathbb{E}\left[\psi(X^{\tilde{u}_j}_{t,x}(T)) \vee \max_{s\in[t,T]} g(X^{\tilde{u}_j}_{t,x}(s))\right] + \varepsilon \\
&\leq \mathbb{E}\left[\Big(\sum_{i\geq 1}\lambda_{ij}\psi(X^{u_{i+j}}_{t,x}(T))\Big) \vee \Big(\max_{s\in[t,T]}\sum_{i\geq 1}\lambda_{ij} g(X^{u_{i+j}}_{t,x}(s))\Big)\right] + \varepsilon \\
&\leq \mathbb{E}\left[\sum_{i\geq 1}\lambda_{ij}\Big(\psi(X^{u_{i+j}}_{t,x}(T)) \vee \max_{s\in[t,T]} g(X^{u_{i+j}}_{t,x}(s))\Big)\right] + \varepsilon \\
&\leq v(t,x) + 2\varepsilon.
\end{aligned}$$

Thanks to the fact that ε is arbitrary we can finally conclude that $\bar{u} \in \mathcal{U}$ is optimal. □

3.10 Appendix: On the proof of the Comparison principle

In this section we provide an extension of the comparison result in [93] to parabolic equations in possibly unbounded domains. Let us denote by e_k the k-th vector of the canonical basis of $\mathbb{R}^n$ $(n \geq 2)$. We will consider a general HJ equation with oblique derivative boundary condition in the direction $-e_n$:

$$\begin{cases} -\partial_t w + H(t,x,Dw,D^2w) = 0 & (0,T) \times D \\ -\partial_{x_n} w = 0 & (0,T) \times \partial D \\ w(T,x) = w_0(x) & \overline{D} \end{cases} \tag{3.10.1}$$

where $D \subseteq \mathbb{R}^n$, $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times S^n \to \mathbb{R}$ and $w_0 : \mathbb{R}^n \to \mathbb{R}$ satisfy the following properties: direction

(P_1) $\overline{D}$ is a locally compact set and $\exists \eta > 0$ such that

$$\bigcup_{0 \leq t \leq \eta} B(x - te_n, t\eta) \subset D^C, \qquad \forall x \in \partial D,$$
$$\bigcup_{0 \leq t \leq \eta} B(x + te_n, t\eta) \subset \overline{D}, \qquad \forall x \in \partial D;$$

(P_2) $H \in C(\mathbb{R} \times \overline{D} \times \mathbb{R}^n \times S^n)$ and there is a neighborhood $\mathcal{O}(\partial D)$ of ∂D in $\overline{D}$ and a function $\omega_1 \in C([0,\infty])$ satisfying $\omega_1(0) = 0$ such that $\forall t \in [0,T),\ x \in \mathcal{O}(\partial D),\ p,q \in \mathbb{R}^n,\ X,Y \in \mathcal{S}^n$:

$$|H(t,x,p,X) - H(t,x,q,Y)| \leq \omega_1(|p-q| + \|X-Y\|);$$

(P_3) There is a function $\omega_2 \in C([0,\infty])$ satisfying $\omega_2(0) = 0$ such that

$$H(t,y,p,-Y) - H(t,x,p,X) \leq \omega_2(\alpha\,|x-y|^2 + |x-y|\,(|p|+1))$$

$\forall \alpha \geq 1, t \in [0,T),\ x,y \in \overline{D},\ p \in \mathbb{R}^n,\ X,Y \in \mathcal{S}^n$ such that

$$-\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Equation (3.5.5) is a particular case of (3.10.1), with $n = d+1$, $\overline{D} \equiv \mathrm{Epigraph}(g)$ and H given by (3.5.6). Under assumption (H_g) the set defined by $\overline{D} = \mathrm{Epigraph}(g)$ satisfies the property (P_1), with $\eta := 1/\sqrt{1+L_g^2}$. The properties (P_2)-(P_3) are also satisfied under $(H'_{b,\sigma})$.

In the sequel we will use the concept of parabolic semijet as defined in Chapter 2, Definition 2.4.6.

Theorem 3.10.1. *Assume (P_1)-(P_3) hold. Let $\underline{w}$ (resp. $\overline{w}$) be a bounded USC viscosity sub-solution (resp. a bounded LSC viscosity super-solution) of (3.10.1). Then $\underline{w} \leq \overline{w}$ on $(0,T] \times \overline{D}$.*

Before starting the proof we state some important preliminary results.

Lemma 3.10.2. *Let (P_1) be satisfied for some $\eta \in (0,1)$. Let us consider the open cone $\Gamma := \bigcup_{t>0} B(-te_n, t\eta)^\circ$. There exists a family $\{w_\varepsilon\}_{\varepsilon>0}$ of C^2 functions on $\mathbb{R}^n \times \mathbb{R}^n$ and* *positive constants* θ, C *such that*

$$
\begin{aligned}
&(3.10.2) \qquad & w_\varepsilon(x,x) &\leq \varepsilon \\
&(3.10.3) \qquad & w_\varepsilon(x,y) &\geq \theta\frac{|x-y|^2}{\varepsilon} \\
&(3.10.4) \qquad & -\langle D_x w_\varepsilon(x,y), e_n\rangle &\geq -C\frac{|x-y|^2}{\varepsilon} \quad \text{if } y-x\notin\Gamma \\
&(3.10.5) \qquad & -\langle D_y w_\varepsilon(x,y), e_n\rangle &\geq 0 \quad \text{if } x-y\notin\Gamma \\
&(3.10.6) \qquad & |D_y w_\varepsilon(x,y)| &\leq C\frac{|x-y|}{\varepsilon}, \\
&(3.10.7) \qquad & |D_x w_\varepsilon(x,y) + D_y w_\varepsilon(x,y)| &\leq C\frac{|x-y|^2}{\varepsilon}
\end{aligned}
$$

and

$$
(3.10.8) \qquad D^2 w_\varepsilon(x,y) \leq C\left\{\frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I\end{pmatrix} + \frac{|x-y|^2}{\varepsilon} I_{2n}\right\}
$$

for $\varepsilon > 0$ *and* $x, y \in \mathbb{R}^n$.

Proof. The result is presented in [93, Theorem 4.1]. □

Lemma 3.10.3. *If* (P_1) *holds, then there exists* $h \in C^2(\overline{D})$ *such that:*

$$
h \geq 0 \quad on\ \overline{D}, \qquad h = 0 \quad in\ \overline{D}\setminus\mathcal{O}(\partial D)
$$

(where $\mathcal{O}(\partial D)$ *is a neighborhood of* ∂D *as in property* (P_2)*) and*

$$
-\partial_{x_n} h(x) \geq 1 \qquad \forall x \in \partial D.
$$

Moreover there exists $M \geq 0$ *such that*

$$
\max_{\overline{D}}(|Dh|, \|D^2 h\|) \leq M.
$$

Proof. Let us consider a point $z \in \partial D$. By hypothesis (P_1) there exists $0 < \delta < 1$ such that

$$
\bigcup_{t\geq 0} B(x - te_n, t\delta) \cap B(z,\delta) \subset D^C;
$$
$$
\bigcup_{t\geq 0} B(x + te_n, t\delta) \cap B(z,\delta) \subset \overline{D}.
$$

We can clearly assume that δ is small enough to have $B(z,\delta)\cap\overline{D} \subset \mathcal{O}(\partial D)$.
We define the hyperplane affine:

$$
H_z := z + \{x \in \mathbb{R}^n : x_n = 0\}.
$$

Let be given a function $\zeta_0 \in C^2(H_z)$ with $\zeta_0(z) = 1$, $\zeta_0 \geq 0$ and supp $\zeta_0 \subset B(z, \delta^2/4) \cap H_z$. We define

$$
\zeta(x) := \zeta_0(x_1, \ldots, x_n)
$$

solution of the following Cauchy problem:

$$\begin{cases} \partial_{x_n}\zeta = 0, \\ \zeta|_{H_z} = \zeta_0. \end{cases} \tag{3.10.9}$$

One has

$$\text{supp }\zeta \subset \bigcup_{t\in\mathbb{R}} B(z - te_n, \frac{\delta^2}{3}).$$

By a geometric consideration we see that there is a small $\varepsilon > 0$ such that

$$\bigcup_{t\in\mathbb{R}} B(z - te_n, \frac{\delta^2}{3}) \cap B(z,\delta) \setminus B(z, \delta-\varepsilon) \subset \bigcup_{t\in\mathbb{R}} B(z - te_n, t\delta)^\circ.$$

Therefore

$$\partial D \cap \text{ supp }\zeta \setminus B(z, \delta - \varepsilon) = \emptyset. \tag{3.10.10}$$

We choose now a C^2 function $\xi \in C_0^2(B(z,\delta)^\circ)$ so that $\xi(x) = 1$ for $x \in B(z, \delta-\varepsilon)$ and $\xi \geq 0$. Let

$$v_z(x) := \zeta(x)\xi(x) \qquad \text{for } x \in B(z,\delta)^\circ.$$

We have that $v_z \in C_0^2(B(z,\delta)^\circ)$, $v_z \geq 0$ and

$$\partial_{x_n} v_z(x) = \partial_{x_n}(\zeta(x)\xi(x)) = \xi(x)\partial_{x_n}\zeta(x) + \zeta(x)\partial_{x_n}\xi(x) = 0$$

if $x \in \partial D$, because ζ is solution of (3.10.9) and moreover $\zeta(x) = 0$ if $x \in \partial D \setminus B(z, \delta-\varepsilon)$ for (3.10.10).
Define now $w_z \in C^2(B(z,\delta))$

$$w_z(x) := z_n - x_n + C$$

with C is a positive constant such that $w_z \geq 0$.
Setting

$$h_z(x) := v_z(x)w_z(x) \qquad \text{for } x \in B(z,\delta)^\circ$$

we find that $h_z \in C^2(B(z,\delta)^\circ)$, $h_z \geq 0$ on $B(z,\delta)^\circ$,

$$\begin{aligned} -\partial_{x_n} h_z(z) &= -\partial_{x_n}(v_z w_z)(z) \\ &= -w_z(z)\partial_{x_n} v_z(z) - v_z(z)\partial_{x_n} w_z(z) \\ &= 0 + v_z(z) = 1 \end{aligned}$$

because $v_z(z) = \zeta_0(z)\xi(z) = 1$. Moreover

$$-\partial_{x_n} h_z(x) = -w_z(x)\partial_{x_n} v_z(x) + v_z(x) = v_z(x) \geq 0$$

for $x \in B(z,\delta)^\circ$ and $-\partial_{x_n} h_z(x) = 0$ on $\partial B(z,\delta)$. We can also observe that the derivatives of h_z are bounded (with a bound depending only on δ). The desired function h can be obtained extending this local construction to the whole boundary of D. In fact there exists $\delta_{1/2} > 0$ such that

$$-\partial_{x_n} h_z(x) \geq \frac{1}{2}, \qquad \forall x \in B(z, \delta_{1/2}) \cap \partial D.$$

Hence, it is possible to cover ∂D with a sequence of balls $B(z_i, \delta)$ choosing the points $z_i \in \partial D, i \in \mathbb{Z}$ such that

$$h(x) := \sum_{i\in\mathbb{Z}} 2h_{z_i}(x)$$

satisfies the desired properties. □

Proof of Theorem 3.10.1. We will prove the theorem for $\underline{w}$ and $\overline{w}$ sub- and super-solution of (3.10.1) with boundary condition respectively replaced by $-\partial_{x_n}\underline{w}+\alpha$ and $-\partial_{x_n}\overline{w}-\alpha$ for a certain $\alpha>0$. It means that on $(0,T)\times\partial D$, $\underline{w}$ and $\overline{w}$ satisfy

$$\min\left(-p_n+\alpha\ ,\ -a+H(t,x,p,X)\right)\le 0\quad \forall (a,p,X)\in\overline{\mathcal{P}}^{1,2,+}\underline{w}(t,x) \tag{3.10.11}$$

$$\max\left(-q_n-\alpha\ ,\ -b+H(t,y,q,Y)\right)\ge 0\quad \forall (b,q,Y)\in\overline{\mathcal{P}}^{1,2,-}\overline{w}(t,y). \tag{3.10.12}$$

We observe that it is always possible to consider a sub-solution $\underline{w}_\gamma$ such that

$$\begin{cases}\lim\limits_{t\to 0}\underline{w}_\gamma(t,x)=-\infty\\ -\partial_t(\underline{w}_\gamma)+H(t,x,D\underline{w}_\gamma,D^2\underline{w}_\gamma)\le -c<0\end{cases}$$

defining, for instance, $\underline{w}_\gamma(t,x):=\underline{w}(t,x)-\frac{\gamma}{t}$. The desired comparison result is then obtained as a limit for $\gamma\to 0$. For simplicity we will still denote $\underline{w}$ such a sub-solution. Given $\beta>0$, let us define $M_\beta:=\sup\limits_{(t,x)\in(0,T]\times\overline{D}}\left(\underline{w}(t,x)-\overline{w}(t,x)-2\beta(1+|x|^2)\right)$. The boundedness of $\underline{w}$ and $\overline{w}$ implies that there exists $(s,z):=(s_\beta,z_\beta)\in(0,T]\times\overline{D}$ such that

$$M_\beta=\underline{w}(s,z)-\overline{w}(s,z)-2\beta(1+|z|^2).$$

If there exists a sequence β_k such that $M_{\beta_k}\le 0$, since $\underline{w}(t,x)-\overline{w}(t,x)\le 2\beta_k(1+|x|^2)+M_{\beta_k}$, for every $(t,x)\in(0,T]\times\overline{D}$, we would have as β_k goes to zero that $\underline{w}(t,x)\le\overline{w}(t,x)$ on $(0,T]\times\overline{D}$. So from now on we will assume that β is small enough such that $M_\beta>0$ and we will show a contradiction. A first consequence is that, thanks to the boundedness of $\underline{w}$ and $\overline{w}$ (let us say that $|\underline{w}|,|\overline{w}|\le M$), one has $\beta|z|^2\le 2M$ and then

$$\beta|z|\to 0\qquad\text{as }\beta\to 0.$$

Moreover, applying standard comparison arguments we can also assume that $z\in\partial D$. Thanks to (P_1), if $z\in\partial D$, there exists $\delta\in(0,\frac{1}{2})$ such that $B(x-te_n,t\delta)\subset D^C$ for $x\in B(z,\delta)\cap\partial D$, $t\in(0,2\delta]$. Set $\Gamma:=\bigcup\limits_{t>0}B(-te_n,t\delta)^\circ$. It comes that

$$y-x\notin\Gamma,\qquad \text{if } x\in\partial D\cap B(z,\delta)^\circ,\ y\in\overline{D}\cap B(z,\delta)^\circ. \tag{3.10.13}$$

In what follows we will restrict our attention to the events on the set $B(z,\delta)\cap\overline{D}$. Thanks to Lemma 3.10.2 we can define

$$\Phi(t,x,y):=\underline{w}(t,x)-\overline{w}(t,y)-\alpha|x-z|^2-w_\varepsilon(x,y)-\beta(1+|x|^2)-\beta(1+|y|^2)-|t-s|^2$$

and thanks to the boundedness and the semicontinuity of $\underline{w}$ and $\overline{w}$ we can state that there exists $(\bar t,\bar x,\bar y):=(\bar t_{\varepsilon,\alpha},\bar x_{\varepsilon,\alpha},\bar y_{\varepsilon,\alpha})\in[0,T)\times\overline{D}\times\overline{D}$ maximum point for Φ. Thanks to (3.10.2) and (3.10.3) the following inequalities hold

(3.10.14)

$$\begin{aligned}M_\beta-\varepsilon&\le\Phi(s,z,z)\le\Phi(\bar t,\bar x,\bar y)\\ &\le\underline{w}(\bar t,\bar x)-\overline{w}(\bar t,\bar y)-\alpha|\bar x-z|^2-\theta\frac{|\bar x-\bar y|^2}{\varepsilon}-\beta(1+|\bar x|^2)-\beta(1+|\bar y|^2)-|\bar t-s|^2\end{aligned}$$

and by classical arguments (see [84, Proposition 3.7]), extracting a subsequence if necessary, we can conclude that as $\varepsilon\to 0$ one has

$$\frac{|\bar x-\bar y|^2}{\varepsilon}\longrightarrow 0\quad\text{and}\quad \bar x,\bar y\longrightarrow z,\quad \bar t\to s.$$

We will take ε small enough such that $\bar{x}, \bar{y} \in B(z, \delta)$.
If $\bar{t} = T$:

$$M_\beta - \varepsilon \leq \Phi(T, \bar{x}, \bar{y}) \leq w_0(\bar{x}) - w_0(\bar{y})$$

so since the right-hand term in the inequality tends to zero for $\varepsilon \to 0$ and $\lim_{\varepsilon \to 0}(M_\beta - \varepsilon) > 0$, we can also assume that $\bar{t} < T$ for ε small enough.
Let us define

$$\begin{aligned}
\underline{v}(t,x) &:= \underline{w}(t,x) - \alpha|x-z|^2 - \beta(1+|x|^2) - |t-s|^2 \\
\overline{v}(t,x) &:= \overline{w}(t,x) + \beta(1+|x|^2).
\end{aligned}$$

Since $(\bar{t}, \bar{x}, \bar{y})$ is a maximum point for Φ and property (3.10.8) holds, for x and y sufficiently close to $\bar{x}$ and $\bar{y}$, we have the following inequalities

$$\begin{aligned}
&\underline{v}(t,x) - \overline{v}(t,y) \\
&\leq \underline{v}(\bar{t},\bar{x}) - \overline{v}(\bar{t},\bar{y}) + w_\varepsilon(x,y) - w_\varepsilon(\bar{x},\bar{y}) \\
&\leq \underline{v}(\bar{t},\bar{x}) - \overline{v}(\bar{t},\bar{y}) + p\cdot(x-\bar{x}) + q\cdot(y-\bar{y}) + \\
&\qquad + \frac{1}{2} D^2 w_\varepsilon(\bar{x},\bar{y}) \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix}, \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} + O(|x-\bar{x}|^3 + |y-\bar{y}|^3) \\
&\leq \underline{v}(\bar{t},\bar{x}) - \overline{v}(\bar{t},\bar{y}) + p\cdot(x-\bar{x}) + q\cdot(y-\bar{y}) \\
&\qquad + \frac{C}{2}\left\{ \frac{1}{\varepsilon} \left\langle \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix}, \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} \right\rangle \right. \\
&\qquad \left. + \left\langle \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix}, \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} \right\rangle \right\} + O(|x-\bar{x}|^3 + |y-\bar{y}|^3) \\
&\leq \underline{v}(\bar{t},\bar{x}) - \overline{v}(\bar{t},\bar{y}) + p\cdot(x-\bar{x}) + q\cdot(y-\bar{y}) + \\
&\qquad + \frac{C}{2}\left\{ \frac{1}{\varepsilon}|(x-\bar{x}) - (y-\bar{y})|^2 + \lambda|x-\bar{x}|^2 + \lambda|y-\bar{y}|^2 \right\},
\end{aligned}$$

with $p = D_x w_\varepsilon(\bar{x},\bar{y})$, $q = D_y w_\varepsilon(\bar{x},\bar{y})$ and $\lambda = \frac{|\bar{x}-\bar{y}|^2}{\varepsilon} + \varepsilon$.
As a consequence of the Crandall-Ishii Lemma (Chapter 2, Lemma 2.4.7) it follows that there exist $\tilde{X}, \tilde{Y} \in S^n$ and $a, b \in \mathbb{R}$ such that

$$-\frac{C}{\varepsilon} I_{2n} \leq \begin{pmatrix} \tilde{X} - C\lambda I & 0 \\ 0 & \tilde{Y} - C\lambda I \end{pmatrix} \leq \frac{C}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{3.10.15}$$

$$a + b = 0 \tag{3.10.16}$$

and

$$(a, D_x w_\varepsilon(\bar{x},\bar{y}), \tilde{X}) \in \overline{\mathcal{P}}^{1,2,+} \underline{v}(\bar{t},\bar{x}) \qquad (-b, -D_y w_\varepsilon(\bar{x},\bar{y}), -\tilde{Y}) \in \overline{\mathcal{P}}^{1,2,-} \overline{v}(\bar{t},\bar{y}).$$

Recalling the definition of $\underline{v}$ and $\overline{v}$ one has

$$(a + 2(\bar{t}-s), D_x w_\varepsilon(\bar{x},\bar{y}) + 2\alpha(\bar{x}-z) + 2\beta\bar{x}, \tilde{X} + 2\alpha I + 2\beta I) \in \overline{\mathcal{P}}^{1,2,+} \underline{w}(\bar{t},\bar{x})$$

and

$$(-b, -D_y w_\varepsilon(\bar{x},\bar{y}) - 2\beta\bar{y}, -\tilde{Y} - 2\beta I) \in \overline{\mathcal{P}}^{1,2,-} \overline{w}(\bar{t},\bar{y}),$$

so just setting $X := \tilde{X} + 2\beta I$ and $Y := Y + 2\beta I$ we get

$$(a + 2(\bar{t} - s), D_x w_\varepsilon(\bar{x}, \bar{y}) + 2\alpha(\bar{x} - z) + 2\beta\bar{x}, X + 2\alpha I) \in \overline{\mathcal{P}}^{1,2,+}\underline{w}(\bar{t}, \bar{x})$$

and

$$(-b, -D_y w_\varepsilon(\bar{x}, \bar{y}) - 2\beta\bar{y}, -Y) \in \overline{\mathcal{P}}^{1,2,-}\overline{w}(\bar{t}, \bar{y}),$$

with

$$-\frac{C}{\varepsilon} I_{2n} \leq \begin{pmatrix} X - C\lambda I - 2\beta I & 0 \\ 0 & Y - C\lambda I - 2\beta I \end{pmatrix} \leq \frac{C}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.10.17}$$

Let us assume that

$$-\langle D_x w_\varepsilon(\bar{x}, \bar{y}) + 2\alpha(\bar{x} - z) + 2\beta\bar{x}, e_n\rangle + \alpha > 0 \tag{3.10.18}$$

$$-\langle -D_y w_\varepsilon(\bar{x}, \bar{y}) - 2\beta\bar{y}, e_n\rangle - \alpha < 0, \tag{3.10.19}$$

then by the definition of viscosity sub- and super-solution of (3.10.1) we get

$$-a - 2(\bar{t} - s) + H(\bar{t}, \bar{x}, D_x w_\varepsilon(\bar{x}, \bar{y}) + 2\alpha(\bar{x} - z) + 2\beta\bar{x}, X + 2\alpha I) < -c < 0$$

and

$$b + H(\bar{t}, \bar{y}, -D_y w_\varepsilon(\bar{x}, \bar{y}) - 2\beta\bar{y}, -Y) \geq 0$$

and from (P_2),(P_3), (3.10.16) and (3.10.17), it follows

$$\begin{aligned}
c &\leq (a + b) + 2(\bar{t} - s) + H(\bar{t}, \bar{y}, -D_y w_\varepsilon(\bar{x}, \bar{y}) - 2\beta\bar{y}, -Y) \\
&\quad - H(\bar{t}, \bar{x}, D_x w_\varepsilon(\bar{x}, \bar{y}) + 2\alpha(\bar{x} - z) + 2\beta\bar{x}, X + 2\alpha I) \\
&\leq H(\bar{t}, \bar{y}, -D_y w_\varepsilon, -Y + C\lambda I + 2\beta I) - H(\bar{t}, \bar{x}, -D_y w_\varepsilon, X - C\lambda I - 2\beta I) \\
&\quad + 2|\bar{t} - s| + \omega_1(|D_x w_\varepsilon + D_y w_\varepsilon| + 2\alpha + C\lambda + 2\beta(2 + |\bar{x}| + |\bar{y}|) + 2\alpha|\bar{x} - z|) \\
&\leq \omega_1(|D_x w_\varepsilon + D_y w_\varepsilon| + 2\alpha + C\lambda + 2\beta(2 + |\bar{x}| + |\bar{y}|) + 2\alpha|\bar{x} - z|) + \\
&\quad + \omega_2(C\frac{|\bar{x} - \bar{y}|^2}{\varepsilon} + |\bar{x} - \bar{y}|(2\beta|\bar{y}| + 1)) + 2|\bar{t} - s|.
\end{aligned}$$

By taking the limit for $\varepsilon \to 0$ we get

$$c \leq \omega_1(2\alpha + 4\beta(1 + |z|))$$

and for α and β small enough we obtain the desired contradiction.
In order to prove (3.10.18) and (3.10.19) we proceed as in [93]. In fact for ε and β sufficiently small

$$-\langle D_x w_\varepsilon(\bar{x}, \bar{y}), e_n\rangle \geq -\frac{\alpha}{2} \Rightarrow -\langle D_x w_\varepsilon(\bar{x}, \bar{y}), e_n\rangle - 2\alpha(\bar{x}_n - z_n) - 2\beta\bar{x}_n + \alpha > 0$$

$$-\langle D_y w_\varepsilon(\bar{x}, \bar{y}), e_n\rangle \geq -\frac{\alpha}{2} \Rightarrow \langle D_y w_\varepsilon(\bar{x}, \bar{y}), e_n\rangle + 2\beta\bar{y}_n - \alpha < 0,$$

then thanks to (3.10.4) and (3.10.5), (3.10.18) and (3.10.19) are finally obtained for ε small enough.
In order to conclude the proof we need to argue how it is possible to modify our boundary conditions.
Let us assume that $\underline{w}$ and $\overline{w}$ are respectively viscosity sub- and super-solution of the original problem. Thanks to Lemma 3.10.3 we can define

$$\underline{w}_\alpha(t, x) := \underline{w}(t, x) - \alpha h(x) - C(T - t)$$

and

$$\overline{w}_\alpha(t,x) := \overline{w}(t,x) + \alpha h(x) + C(T-t).$$

Let $(a,p,X) \in \overline{\mathcal{P}}^{1,2,+}\underline{w}_\alpha(t,x)$ and $(b,q,Y) \in \overline{\mathcal{P}}^{1,2,-}\overline{w}_\alpha(t,x)$. By using ($P_2$), one has

$$-a + H(t,x,p,X) \le -a + C + H(t,x,p+\alpha Dh, X + \alpha D^2 h) - C + \omega_1(\alpha M), \tag{3.10.20}$$

$$-b + H(t,x,q,Y) \ge -b - C + H(t,x,q-\alpha Dh, Y - \alpha D^2 h) + C - \omega_1(\alpha M) \tag{3.10.21}$$

where $M := \max_{\overline{D}}(|Dh| + \|D^2 h\|)$ and for any $x \in \partial D$

$$\begin{aligned} -\langle p, e_n\rangle &= -\langle p + \alpha Dh(x), e_n\rangle + \alpha \partial_{x_n} h(x) \\ &\le \langle p + \alpha Dh(x), e_n\rangle - \alpha \end{aligned} \tag{3.10.22}$$

$$-\langle q, e_n\rangle \ge -\langle q - \alpha Dh(x), e_n\rangle + \alpha. \tag{3.10.23}$$

Observing that

$$\overline{\mathcal{P}}^{1,2,+}\underline{w}(t,x) = \overline{\mathcal{P}}^{1,2,+}\underline{w}_\alpha(t,x) + \left(-C, \alpha Dh, \alpha D^2 h\right)$$
$$\overline{\mathcal{P}}^{1,2,-}\overline{w}(t,x) = \overline{\mathcal{P}}^{1,2,-}\overline{w}_\alpha(t,x) - \left(-C, \alpha Dh, \alpha D^2 h\right)$$

one has

$$\left(a - C, p + \alpha Dh(x), X + \alpha D^2 h(x)\right) \in \overline{\mathcal{P}}^{1,2,+}\underline{w}(x)$$
$$\left(b + C, q - \alpha Dh(x), Y - \alpha D^2 h(x)\right) \in \overline{\mathcal{P}}^{1,2,-}\overline{w}(x)$$

then, by the definition of viscosity sub- and super-solution

$$\begin{cases} -a + C + H(t,x,p+\alpha Dh, X + \alpha D^2 h) \le 0 & \text{on } D \\ \min\left(-\langle p + \alpha Dh(x), e_n\rangle, -a + C + H(t,x,p+\alpha Dh, X+\alpha D^2 h)\right) \le 0 & \text{on } \partial D \end{cases}$$

and

$$\begin{cases} -b - C + H(t,x,q-\alpha Dh, Y - \alpha D^2 h) \ge 0 & \text{on } D \\ \max\left(-\langle q - \alpha Dh(x), e_n\rangle, -b - C + H(t,x,q-\alpha Dh, Y-\alpha D^2 h)\right) \ge 0 & \text{on } \partial D. \end{cases}$$

For α small enough, taking $C = w_1(\alpha M)$, we can finally conclude by inequalities (3.10.20),(3.10.22) and (3.10.21),(3.10.23) that

$$\begin{cases} -a + H(t,x,p,X) \le 0 & \text{on } D \\ \min\left(-p_n + \alpha, -a + H(t,x,p,X)\right) \le 0 & \text{on } \partial D \end{cases}$$

and

$$\begin{cases} b + H(t,x,q,Y) \ge 0 & \text{on } D \\ \max\left(-q_n - \alpha, b + H(t,x,q,Y)\right) \ge 0 & \text{on } \partial D. \end{cases}$$

In other words for α small enough $\underline{w}_\alpha$ and $\overline{w}_\alpha$ are respectively sub- and super-solution of (3.10.1) with boundary conditions $-\partial_{x_n}\underline{w}_\alpha + \alpha$ and $-\partial_{x_n}\overline{w}_\alpha - \alpha$ and since $\underline{w}_\alpha \to \underline{w}$ and $\overline{w}_\alpha \to \overline{w}$ as α goes to 0, we are allowed to prove the comparison theorem for them instead of $\underline{w}$ and $\overline{w}$. □

Chapter 4

Zubov's method for controlled diffusions with state constraints

Publications of this chapter:

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4.1 Introduction

In this chapter we aim to study the asymptotic controllability property of controlled stochastic systems in presence of state constraints.

The basic problem in this context is the existence of a control strategy that asymptotically steers the system to a certain target set with positive probability. In the uncontrolled framework, the idea, due to Lyapunov, of linking the stability properties of a system with the existence of a continuous function (in the nowadays literature called a "Lyapunov function") that decreases along the trajectories of the system, represents a fundamental tool for the study of this kind of problems. In his seminal thesis [137], Lyapunov proved that the existence of such a function is a sufficient condition for the asymptotic stability around a point of equilibrium of a dynamical system

$$\dot{x} = b(x), \qquad x(t) \in \mathbb{R}^d, t \geq 0. \tag{4.1.1}$$

This theory was further developed in the later years, see [105, 140, 123], and also also the converse property was established. Since the 60s, Lyapunov's method was extended to stochastic diffusion processes. The main contributions in this framework come from [106, 125, 124, 126], where the concepts of stability and asymptotic stability in probability, as well as the stronger concept of almost sure stability, are introduced.

An important domain of research concerns the developments of constructive procedure for the definition of Lyapunov functions. In the deterministic case an important result was obtained by Zubov in [174]. In this work the domain attraction of an equilibrium point $x_E \in \mathbb{R}^d$ for the system (4.1.1), i.e. the set of initial points that are asymptotically attracted by x_E, is characterized by using the solution ϑ of the following first order equation

$$\begin{cases} D\vartheta(x)b(x) = -f(x)(1-\vartheta(x))\sqrt{1+\|b(x)\|^2} & x \in \mathbb{R}^d \setminus \{x_E\} \\ \vartheta(x_E) = 0, \end{cases} \tag{4.1.2}$$

for a suitable choice of a scalar function f (see [174] and [105]). Equation (4.1.2) is referred to in the literature as Zubov equation. In particular what is proved in [174] is that the domain of attraction coincides with the set of points $x \in \mathbb{R}^d$ such that $\vartheta(x) < 1$. Further developments and applications of this method can be found in [22, 105, 1, 103, 118, 70].

More recently, this kind of approach has been applied to more general systems, included control systems, thanks also to the advances of the viscosity solution theory that allow to consider merely continuous solutions of fully nonlinear PDE's. While for systems of ordinary differential equations the property of interest is stability, for systems that involve controls, the interest lies on "controllability", i.e. on the existence of a control such that the associated trajectory asymptotically reaches the target represented by the equilibrium point (see [10, 164]). The case of deterministic control systems was considered in [72]. Here, through the formulation of a suitable optimal control problem, it is proved that the domain of attraction can be characterized by the solution of a nonlinear PDE (that we can consider as a generalized Zubov equation) which turns out to be a particular kind of HJB equation. In this case the existence of smooth solutions is not guaranteed and therefore the equation is considered in the viscosity sense. The state constrained case, where we aim to steer the system to the target satisfying at the same time some constraints on the state, has been treated in [104].

The Zubov method has been extended to the stochastic setting in [73] and [69] taking into account diffusion processes. The controlled case was later considered in [71] and [67]. In this last paper, under some property of local exponential stabilizability in probability of the target set (that weakens the "almost sure" stabilizability assumption made in [71] and [73]), the set of points $x \in \mathbb{R}^d$ that can be asymptotically steered with positive probability towards the target, is characterized by means of the unique viscosity solution with value zero on the target of the following equation

$$\sup_{u\in U}\Big\{-f(x,u)(1-\vartheta(x)) - D\vartheta(x)b(x,u)\frac{1}{2}Tr[\sigma\sigma^T(x,u)D^2\vartheta(x)]\Big\} = 0.$$

In this chapter we aim to add state constraints in this framework, trying to exploit the ideas proposed in [104]. By the way the results in terms of PDE characterization of the domain of attraction will be very different. In [104] the state constrained controllability is characterized by the solution of an obstacle problem, whereas in our case we will deal with a mixed Dirichlet-Neumann boundary problem (see Section 4.5).

The chapter is organized as follows: in Section 4.2 we introduce the setting and the main assumptions. Section 4.3 is devoted to the study of some properties of the domain of attraction. In Section 4.4 is defined our level set function v as the value function associated with an optimal control problem with a maximum cost and the domain of attraction is characterized as a sub-level set of v. In Section 4.5 the domain of attraction is characterized by the viscosity solution of second order nonlinear PDE with mixed Dirichlet-Neumann boundary conditions. A comparison principle for bounded viscosity sub- and super-solution of this problem is provided in Section 4.6.

4.2 Setting

Let $(\Omega, \mathcal{F}, \{\mathbb{F}\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space supporting a p-dimensional Brownian motion $\mathcal{B}(\cdot)$, where $\{\mathbb{F}_t\}_{t\geq 0}$ denotes the $\mathbb{P}$-augmentation of the filtration generated by $\mathcal{B}$.

We consider the following system of autonomous SDE's in $\mathbb{R}^d$ $(d \geq 1)$

$$
\begin{cases} dX(t) = b(X(t), u(t))dt + \sigma(X(t), u(t))d\mathcal{B}(t) & t > 0,\ x \in \mathbb{R}^d, \\ X(0) = x \end{cases} \tag{4.2.1}
$$

where $u \in \mathcal{U}$, and $\mathcal{U}$ denotes the set of the $\{\mathbb{F}_t\}$-progressively measurable processes taking values in a compact set $U \subset \mathbb{R}^m$ (that is we will work under the assumption (H_U)).

Under the classical assumptions (H_b) and (H_σ) (with b and σ independent of t in this case), we will denote by $X_x^u(\cdot)$ the unique strong solution of (4.2.1) associated with the control $u \in \mathcal{U}$ and the initial position $x \in \mathbb{R}^d$. By $\mathcal{T} \subset \mathbb{R}^d$ we denote a compact target set for the system, i.e., a set towards which we want to asymptotically drive the trajectories. The set $\mathcal{K} \subseteq \mathbb{R}^d$ represents the state constraints for system (4.2.1), i.e., the open set where we want to maintain the state $X_x(t)$ with a positive probability for all $t \geq 0$, cf. the definition of the set $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ below. We impose the following assumptions on the target and the set of state constraints:

$$
(H'_{\mathcal{T}}) \qquad \begin{cases} (i) \quad \mathcal{T} \text{ is a nonempty and compact set;} \\ (ii) \quad \mathcal{T} \text{ is viable for (4.2.1)} : \forall x \in \mathcal{T} \text{ there is } u \in \mathcal{U} \text{ such that} \\ \qquad\qquad X_x^u(t) \in \mathcal{T} \quad \forall t \geq 0 \qquad \text{a.s.;} \\ (iii) \quad \mathcal{T} \text{ is locally exponentially stabilizable in probability:} \\ \qquad \text{there exist positive constants } r, \lambda \text{ such that for every} \\ \qquad \varepsilon > 0, \exists C_\varepsilon > 0 \text{ such that } \forall x \in \mathcal{T}_r \text{ there is } u \in \mathcal{U} \text{ such that} \\ \mathbb{P}\left[\sup_{t\geq 0} d_{\mathcal{T}}^+(X_x^u(t))e^{\lambda t} \leq C_\varepsilon d_{\mathcal{T}}^+(x) \text{ and } X_x^u(t) \in \mathcal{K}\ \forall t \geq 0\right] \geq 1 - \varepsilon. \end{cases}
$$

$$
(H'_{\mathcal{K}}) \qquad \mathcal{K} \text{ is an open set in } \mathbb{R}^d.
$$

For necessary and sufficient conditions for the viability of $\mathcal{T}$ we remand to Section 2.5 in Chapter 2 and the references therein. For simplicity we assume that $\mathcal{T} \subset \mathcal{K}$. Note that this implies that for r small enough one has $\mathcal{T}_r := \{x \in \mathbb{R}^d : d_{\mathcal{T}}^+(x) \leq r\} \subset \mathcal{K}$, where, as in the previous chapters $d_{\mathcal{T}}^+(\cdot)$ denotes the positive Euclidean distance to $\mathcal{T}$.

Remark 4.2.1. We point out that assumption $(H'_{\mathcal{T}})$ implies that for any $x \in \mathcal{T}_r$

$$
\sup_{u\in\mathcal{U}} \mathbb{P}\left[\lim_{t\to+\infty} d_{\mathcal{T}}^+(X_x^u(t)) = 0 \text{ and } X_x^u(t) \in \mathcal{K}\ \forall t \geq 0\right] = 1.
$$

Indeed, for any $\varepsilon > 0$ and for suitable positive constants λ and C_ε, the local exponentially stabilizability implies the existence of a control $u \in \mathcal{U}$ such that

$$
\begin{aligned}
(1-\varepsilon) &\leq \mathbb{P}\left[\sup_{t\geq 0} d_{\mathcal{T}}^+(X_x^u(t), \mathcal{T})e^{\lambda t} \leq C_\varepsilon d_{\mathcal{T}}^+(x) \text{ and } X_x^u(t) \in \mathcal{K}\ \forall t \geq 0\right] \\
&\leq \mathbb{P}\left[\lim_{t\to+\infty} d_{\mathcal{T}}^+(X_x^u(t)) = 0 \text{ and } X_x^u(t) \in \mathcal{K}\ \forall t \geq 0\right],
\end{aligned}
$$

and the result follows by the arbitrariness of ε. We also note that without loss of generality we may assume that $r > 0$ is so small that $\mathcal{T}_r \subset \mathcal{K}$.

Aim of this chapter is to characterize the set $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ of initial states $x \in \mathbb{R}^d$ which can be driven by an admissible control to the target $\mathcal{T}$ with positive probability:

$$\begin{aligned}\mathcal{D}^{\mathcal{T},\mathcal{K}} := \Big\{ & x \in \mathbb{R}^d : \exists u \in \mathcal{U} \ \text{ s.t.} \\ & \mathbb{P}\Big[\lim_{t\to+\infty} d_{\mathcal{T}}^{+}(X_x^u(t)) = 0 \text{ and } X_x^u(t) \in \mathcal{K}\ \forall t \geq 0 \Big] > 0 \Big\} \\ = \Big\{ & x \in \mathbb{R}^d : \sup_{u\in\mathcal{U}} \mathbb{P}\Big[\lim_{t\to+\infty} d_{\mathcal{T}}^{+}(X_x^u(t)) = 0 \text{ and } X_x^u(t) \in \mathcal{K}\ \forall t \geq 0 \Big] > 0 \Big\}.\end{aligned}$$

The set $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ is called the *domain of asymptotic controllability* (with positive probability) of $\mathcal{T}$.

4.3 Some results on the domain $\mathcal{D}^{\mathcal{T},\mathcal{K}}$

For any $x \in \mathbb{R}^d$ and $u \in \mathcal{U}$ we introduce the random hitting time $\tau(x,u)$ as the first time instant when the trajectory starting at point x and driven by the control u hits the set $\mathcal{T}_r$, that is for any $\omega \in \Omega$

$$\tau(x,u)(\omega) := \inf\big\{ t \geq 0 : X_x^u(t)(\omega) \in \mathcal{T}_r \big\}. \tag{4.3.1}$$

Remark 4.3.1. We will assume that the set of admissible control laws $\mathcal{U}$ satisfies the *stability under concatenation* and *stability under measurable selection* properties. The set $\mathcal{U}$ satisfies the condition of stability under concatenation if for any stopping time τ and any two control processes $u_1, u_2 \in \mathcal{U}$ the τ-concatenation of u_1 and u_2, defined by

$$u_1 \oplus_\tau u_2(\omega,t) := \begin{cases} u_1(\omega,t) & \text{if } t \leq \tau \\ u_2(\omega, t-\tau) & \text{otherwise,} \end{cases}$$

is an admissible control. For the condition of stability under measurable selection we require that for all stopping times τ and all maps $\Phi : \Omega \to \mathcal{U}$, measurable with respect to the corresponding σ-algebras, there exists a $\nu \in \mathcal{U}$ such that

$$\Phi(\omega)(t) = \nu(t) \quad \text{for } Leb \times \mathbb{P}\text{-almost all } (t,\omega) \text{ such that } t \geq \tau(\omega).$$

These two properties guarantee the validity of the Dynamic Programming Principle, Lemma 4.5.1, under standard regularity assumptions on the coefficients of the problem (see [107]). In our context, such properties also play another important role in ensuring the controllability of the system. Indeed, for every $y \in \mathcal{T}_r$ the exponential stabilizability property guarantees the existence of a control $u_y \in \mathcal{U}$ such that $(H'_{\mathcal{T}})(iii)$ holds. Intuitively, this means that once a path associated with a control u hits the boundary of $\mathcal{T}_r$ at time $\tau := \tau(x,u)$, we can control it to $\mathcal{T}$ by switching to the process $u_{X(\tau)} := u_{X_x^u(\tau(x,u))}$. However, this construction is only possible if the process

$$\bar{u}(t) = u(t)\mathbb{1}_{\{t\leq\tau\}} + \Big(u(t)\mathbb{1}_{\{\tau=+\infty\}} + u_{X(\tau)}(t-\tau)\mathbb{1}_{\{\tau<\infty\}} \Big)\mathbb{1}_{\{t>\tau\}} \tag{4.3.2}$$

belongs to $\mathcal{U}$ and, in general, this cannot be guaranteed in our framework. As a remedy the following construction can be used. Let us define the map

$$\bar{\Phi} : \omega \mapsto \begin{cases} u_{X_x^u(\tau(x,u))}(\cdot) & \text{if } \omega \in \{\omega \in \Omega : \tau(x,u)(\omega) < +\infty\} \\ u(\cdot) & \text{otherwise.} \end{cases}$$

We can observe that $\{\omega \in \Omega : \tau(x,u)(\omega) < +\infty\}$ is $\mathcal{F}_\tau$-measurable and then the map $\bar{\Phi}$ is measurable from $(\Omega, \mathcal{F}_\tau)$ to $(\mathcal{U}, \mathcal{B}(\mathcal{U}))$ so, if stability under measurable selection holds, there exists $\nu \in \mathcal{U}$ such that

$$\bar{\Phi}(\omega)(t) = \nu(t) \quad \text{for } Leb \times \mathbb{P}\text{-almost all } (t,\omega) \text{ such that } t \geq \tau(x,u)(\omega).$$

Under the assumption of stability under concatenation, we also have that the control $u \oplus_{\tau(x,u)} \nu$ belongs to $\mathcal{U}$, so that it can be finally used to steer the system to $\mathcal{T}$. With a slight abuse of notation we will still denote in the chapter such a control by means of expression (4.3.2), but the reader should always keep in mind this construction. For a discussion of existence results for control laws satisfying the stability properties just described we refer to [67, Section 2]

Our goal is now to establish a relation between the set $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ and the hitting time $\tau(x,u)$. To this end, we start with the following preliminary result. Therein and in the rest of the chapter we use the notation $X_\tau^u := X_x^u(\tau(x,u))$.

Lemma 4.3.2. *Let assumptions* (H_b),(H_σ),$(H'_\mathcal{T})$ *and* $(H'_\mathcal{K})$ *be satisfied. Then for the hitting time* $\tau(x,u)$ *from* (4.3.1) *there exist positive constants* λ, C *such that*

$$\sup_{u\in\mathcal{U}} \mathbb{P}\Big[\tau(x,u) < +\infty \textit{ and } X_x^u(t) \in \mathcal{K}\ \forall t \in [0,\tau(x,u)]\Big] > 0$$
$$\Rightarrow \sup_{u\in\mathcal{U}} \mathbb{P}\Big[\tau(x,u) < +\infty,\ X_x^u(t) \in \mathcal{K}\ \forall t \geq 0 \textit{ and } \sup_{t\geq 0} d_\mathcal{T}^+(X_{X_\tau^u}^{u(\tau(x,u)+\cdot)}(t))e^{\lambda t} \leq C\Big] > 0.$$

Proof. The statement is proved using the exponential stabilizability assumption. By assumption there exists $\nu \in \mathcal{U}$ such that

$$\mathbb{P}\Big[\tau(x,\nu) < +\infty \text{ and } X_x^\nu(t) \in \mathcal{K}\ \forall t \in [0,\tau(x,\nu)]\Big] > 0.$$

Moreover, thanks to assumption $(H'_\mathcal{T})$-(iii), constants $\lambda, C > 0$ can be found such that for any $y \in \mathcal{T}_r$, there is $u_y \in \mathcal{U}$ with

$$\mathbb{P}\Big[\sup_{t\geq 0} d_\mathcal{T}^+(X_y^{u_y}(t))e^{\lambda t} \leq C \text{ and } X_y^{u_y}(t) \in \mathcal{K}\ \forall t \geq 0\Big] \geq \frac{1}{2}.$$

Therefore, abbreviating $\tau = \tau(x,\nu) = \tau(x,\bar{\nu})$ and defining the control

$$\bar{\nu}(t) := \nu(t)\mathbb{1}_{\{t\leq\tau\}} + \Big(\nu(t)\mathbb{1}_{\{\tau=+\infty\}} + u_{X_\tau^\nu}(t-\tau)\mathbb{1}_{\{\tau<\infty\}}\Big)\mathbb{1}_{\{t>\tau\}},$$

see Remark 4.3.1, one obtains

$$\begin{aligned}
&\mathbb{P}\Big[\tau(x,\bar\nu)<+\infty,\ X_x^{\bar\nu}(t)\in\mathcal{K}\ \forall t\geq 0 \text{ and } \sup_{t\geq 0} d_{\mathcal{T}}^{+}(X_{X_\tau^{\bar\nu}}^{\bar\nu(\tau+\cdot)}(t))e^{\lambda t}\leq C\Big]\\
&=\mathbb{P}\Big[\tau(x,\bar\nu)<+\infty,\ X_x^{\bar\nu}(t)\in\mathcal{K}\ \forall t\in[0,\tau(x,\bar\nu)],\ X_{X_\tau^{\bar\nu}}^{\bar\nu(\tau+\cdot)}(t)\in\mathcal{K}\ \forall t\geq 0\\
&\quad \text{and } \sup_{t\geq 0} d_{\mathcal{T}}^{+}(X_{X_\tau^{\bar\nu}}^{\bar\nu(\tau+\cdot)}(t))e^{\lambda t}\leq C\Big]\\
&=\int_0^{+\infty}\int_{d_{\mathcal{T}}^{+}(y)=r}\mathbb{P}\Big[X_y^{u_y}(t)\in\mathcal{K}\ \forall t\geq 0 \text{ and } \sup_{t\geq 0} d_{\mathcal{T}}^{+}(X_y^{u_y}(t))e^{\lambda t}\leq C\Big| X_s^{\nu}=y\Big]\\
&\quad\cdot\mathbb{P}\Big[X_\tau^{\nu}=dy,\ \tau(x,\nu)=ds \text{ and } X_x^{\nu}(t)\in\mathcal{K}\ \forall t\in[0,\tau(x,\nu)]\Big]\\
&\geq\frac{1}{2}\int_0^{+\infty}\int_{d_{\mathcal{T}}^{+}(y)=r}\mathbb{P}\Big[X_\tau^{\nu}=dy,\ \tau(x,\nu)=ds \text{ and } X_x^{\nu}(t)\in\mathcal{K}\ \forall t\in[0,\tau(x,\nu)]\Big]\\
&=\frac{1}{2}\,\mathbb{P}\Big[\tau(x,\nu)<+\infty \text{ and } X_x^{u}(t)\in\mathcal{K}\ \forall t\in[0,\tau(x,\nu)]\Big]>0.
\end{aligned}$$

□

Thanks to the previous result, the following alternative characterization of $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ is obtained.

Proposition 4.3.3. *Let assumptions* (H_b),(H_σ),$(H'_{\mathcal{T}})$ *and* $(H'_{\mathcal{K}})$ *be satisfied. Then*

$$\mathcal{D}^{\mathcal{T},\mathcal{K}}=\Big\{x\in\mathbb{R}^d:\sup_{u\in\mathcal{U}}\mathbb{P}\Big[\tau(x,u)<+\infty \textit{ and } X_x^u(t)\in\mathcal{K}\ \forall t\in[0,\tau(x,u)]\Big]>0\Big\}.$$

Proof. The "$\subseteq$" inclusion is immediate since for every $u\in\mathcal{U}$ one has

$$\begin{aligned}
&\Big\{\omega\in\Omega:\lim_{t\to+\infty} d_{\mathcal{T}}^{+}(X_x^u(t))=0 \text{ and } X_x^u(t)\in\mathcal{K}\ \forall t\geq 0\Big\}\\
&\subseteq\Big\{\omega\in\Omega:\tau(x,u)<+\infty \ \text{ and }\ X_x^u(t)\in\mathcal{K}\ \forall t\in[0,\tau(x,u)]\Big\}.
\end{aligned}$$

For the converse inclusion, consider $x\in\mathbb{R}^d$ with

$$\sup_{u\in\mathcal{U}}\mathbb{P}\Big[\tau(x,u)<+\infty \text{ and } X_x^u(t)\in\mathcal{K}\ \forall t\in[0,\tau(x,u)]\Big]>0.$$

Then, Lemma 4.3.2 yields

$$\sup_{u\in\mathcal{U}}\mathbb{P}\Big[\tau(x,u)<+\infty,\ X_x^u(t)\in\mathcal{K}\ \forall t\geq 0 \text{ and } \sup_{t\geq 0} d_{\mathcal{T}}^{+}(X_{X_\tau^u}^{u(\tau(x,u)+\cdot)}(t))e^{\lambda t}\leq C\Big]>0$$

which immediately implies

$$\sup_{u\in\mathcal{U}}\mathbb{P}\Big[X_x^u(t)\in\mathcal{K}\ \forall t\geq 0 \text{ and } \lim_{t\to\infty} d_{\mathcal{T}}^{+}(X_{X_\tau^u}^{u(\tau(x,u)+\cdot)}(t))=0\Big]>0$$

and thus $x\in\mathcal{D}^{\mathcal{T},\mathcal{K}}$. □

One also has

Proposition 4.3.4. *Assume assumptions* (H_b),(H_σ),$(H'_{\mathcal{T}})$ *and* $(H'_{\mathcal{K}})$ *be satisfied. Then* $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ *is an open set.*

Proof. Let us start observing that for any $x \in \mathcal{D}^{\mathcal{T},\mathcal{K}}$, there is a time $T > 0$ and a control $\nu \in \mathcal{U}$ such that

$$\mathbb{P}\Big[d^+_{\mathcal{T}}(X^\nu_x(T)) \leq \frac{r}{2} \text{ and } X^\nu_x(t) \in \mathcal{K}\ \forall t \geq 0\Big] =: \eta > 0.$$

Thanks to assumptions (H_b) and (H_σ), one has that for any $\varepsilon > 0$

$$\lim_{|x-y|\to 0} \mathbb{P}\Big[\sup_{s\in[0,T]} \big|X^\nu_x(t) - X^\nu_y(t)\big| > \varepsilon\Big] = 0,$$

therefore we can find $\delta_\eta > 0$ such that for any x, y such that $|x - y| \leq \delta_\eta$

$$\mathbb{P}\Big[\sup_{s\in[0,T]} \big|X^\nu_x(t) - X^\nu_y(t)\big| > \varepsilon\Big] \leq \frac{\eta}{2}.$$

It follows that for any fixed $\varepsilon > 0$ if $y \in B(x, \delta_\eta)$, the set $\Omega_1 \subset \mathcal{F}$ defined by

$$\Omega_1 := \Big\{\omega \in \Omega : d^+_{\mathcal{T}}(X^\nu_x(T)(\omega)) \leq \frac{r}{2},\ X^\nu_x(t)(\omega) \in \mathcal{K}\ \forall t \geq 0 \\ \text{and } \sup_{s\in[0,T]} \big|X^\nu_x(t) - X^\nu_y(t)\big|(\omega) \leq \varepsilon\Big\}$$

satisfies

$$\begin{aligned}
&\mathbb{P}[\Omega_1] \\
&= \mathbb{P}\Big[\Big\{d^+_{\mathcal{T}}(X^{\bar u}_x(T)) \leq \frac{r}{2},\ X^{\bar u}_x(t) \in \mathcal{K}\ \forall t \geq 0\Big\} \cap \Big\{\sup_{s\in[0,T]} \big|X^{\bar u}_x(t) - X^{\bar u}_y(t)\big| \leq \varepsilon\Big\}\Big] \\
&= 1 - \mathbb{P}\Big[\Big\{d^+_{\mathcal{T}}(X^\nu_x(T)) \leq \frac{r}{2},\ X^\nu_x(t) \in \mathcal{K}\ \forall t \geq 0\Big\}^C \cup \Big\{\sup_{s\in[0,T]} \big|X^\nu_x(t) - X^\nu_y(t)\big| > \varepsilon\Big\}\Big] \\
&\geq 1 - \mathbb{P}\Big[\Big\{d^+_{\mathcal{T}}(X^\nu_x(T)) \leq \frac{r}{2},\ X^\nu_x(t) \in \mathcal{K}\ \forall t \geq 0\Big\}^C\Big] - \mathbb{P}\Big[\sup_{s\in[0,T]} \big|X^\nu_x(t) - X^\nu_y(t)\big| > \varepsilon\Big] \\
&\geq 1 - 1 + \eta - \frac{\eta}{2} = \frac{\eta}{2} > 0.
\end{aligned}$$

For any $\omega \in \Omega_1$, since $X^\nu_x(t) \in \mathcal{K}, \forall t \geq 0$ and $\mathcal{K}$ is an open set one has

$$\delta(x, \nu)(\omega) := \inf_{t\in[0,T]} d^+_{\mathcal{K}^C}(X^\nu_x(t))(\omega) > 0.$$

and

$$\sup_{t\in[0,T]} |X^\nu_x(t) - X^\nu_y(t)|(\omega) < \delta(x, \nu)(\omega) \Rightarrow X^\nu_y(t)(\omega) \in \mathcal{K}, \quad \forall t \in [0, T].$$

Furthermore it is also possible to prove that there exist $M > 0$ and $\tilde\Omega_1 \subseteq \Omega_1$ with $\mathbb{P}[\tilde\Omega_1] > 0$ such that

$$\forall \omega \in \tilde\Omega_1 \qquad \delta(x, \nu)(\omega) > M. \tag{4.3.3}$$

Indeed defined

$$B_n := \left\{ \omega \in \Omega_1 : \delta(x,\nu)(\omega) \in [\frac{1}{n+1}, \frac{1}{n}) \right\}$$

one has

$$0 < \mathbb{P}[\Omega_1] = \mathbb{P}[\bigcup_{n\geq 0} B_n] = \sum_{n\geq 0} \mathbb{P}[B_n].$$

It means that there exists $\bar{n} \in \mathbb{N}$ such that $\mathbb{P}[B_{\bar{n}}] > 0$ and defined

$$\tilde{\Omega}_1 \quad := \quad \left\{ \omega \in \Omega_1 : \delta(x,\nu)(\omega) \geq \frac{1}{\bar{n}+1} \right\}$$

we have $\mathbb{P}[\tilde{\Omega}_1] \geq \mathbb{P}[B_{\bar{n}}] > 0$. We have now all the elements necessary for concluding the proof. Taking $\varepsilon \leq \min\{M/2, r/2\}$ we have that for any $\omega \in \tilde{\Omega}_1$

$$X_y^\nu(t)(\omega) \in \mathcal{K}, \forall t \in [0,T]$$

and

$$d_{\mathcal{T}}^+(X_y^\nu(T))(\omega) \leq d_{\mathcal{T}}^+(X_x^\nu(T))(\omega) + |X_x^\nu(T) - X_y^\nu(T)|(\omega) \leq \frac{r}{2} + \varepsilon \leq r$$

that is $\tau(y,\nu)(\omega) \leq T$.
In conclusion we have proved that there exists a control $\nu \in \mathcal{U}$ such that for any $y \in B(x, \delta_\eta)$

$$\mathbb{P}\Big[\tau(y,\nu) < +\infty \quad \text{and} \quad X_y^\nu(t) \in \mathcal{K}\ \forall t \in [0, \tau(y,\nu)]\Big] > 0,$$

that means $y \in \mathcal{D}^{\mathcal{T},\mathcal{K}}$. □

4.4 The level set function v

Similarly to what we did in Chapter 3 for characterizing the backward reachable set $\mathcal{R}_t^{\mathcal{T},\mathcal{K}}$, we are now going to define a function v that we will use in order to characterize the domain $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ as a sub-level set. Let us start introducing two functions $f : \mathbb{R}^d \times U \to \mathbb{R}$ and $h : \mathbb{R}^d \to [0, +\infty]$ such that

$$(H_f) \qquad \begin{cases} (i)\ f(\cdot,\cdot) \text{ is continuous on } \mathbb{R}^d \times U; \\ (ii) \text{ there exist constants } L_f,\ M_f \text{ and } f_0 > 0 \text{ such that} \\ \qquad |f(x,u) - f(x',u)| \leq L_f|x - x'|; \\ \qquad f(x,u) \leq M_f; \\ \qquad f \geq 0 \quad \text{and} \quad f(x,u) = 0 \Leftrightarrow x \in \mathcal{T}; \\ \qquad \inf_{u\in U} f(x,u) \geq f_0 > 0, \quad \forall x \in \mathbb{R}^d \setminus \mathcal{T}_r \\ \quad \text{for any } x, x' \in \mathbb{R}^d,\ u \in U \text{ and } \mathcal{T}, \mathcal{T}_r \text{ from } (H'_{\mathcal{T}}). \end{cases}$$

$$(H_h) \qquad \begin{cases} \quad h \text{ is a locally Lipschitz continuous function in } \mathcal{K} \text{ such that ;} \\ (i) \quad h(x) = +\infty \Leftrightarrow x \notin \mathcal{K}; \\ \qquad h(x_n) \to +\infty, \quad \forall x_n \to x \notin \mathcal{K}; \\ \qquad h(x) = 0, \quad \forall x \in \mathcal{T}; \\ (ii) \text{ there exists a constant } L_h \geq 0 \text{ such that} \\ \qquad \left| e^{-h(x)} - e^{-h(x')} \right| \leq L_h|x - x'| \\ \quad \text{for any } x, x' \in \mathbb{R}^d. \end{cases}$$

Let the function $v : \mathbb{R}^d \to [0,1]$ be defined by:

$$v(x) := \inf_{u\in\mathcal{U}} \Big\{ 1 + \mathbb{E}\Big[\sup_{t\geq 0}\Big(- e^{-\int_0^t f(X_x^u(s),u(s))ds - h(X_x^u(t))}\Big)\Big]\Big\}. \tag{4.4.1}$$

We will now show that the function v can be used in order to characterize the domain of controllability $\mathcal{D}^{\mathcal{T},\mathcal{K}}$. In particular, we are going to prove that $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ consists of the set of points x where v is strictly lower than one.

Theorem 4.4.1. *Let assumptions* (H_b),(H_σ),$(H'_{\mathcal{T}})$,$(H'_{\mathcal{K}})$,(H_f) *and* (H_h) *be satisfied, then*

$$x \in \mathcal{D}^{\mathcal{T},\mathcal{K}} \Leftrightarrow v(x) < 1.$$

Proof. "$\Leftarrow$" We show $v(x) = 1$ for every $x \notin \mathcal{D}^{\mathcal{T},\mathcal{K}}$. If $x \notin \mathcal{D}^{\mathcal{T},\mathcal{K}}$ then Proposition 4.3.3 implies

$$\sup_{u\in\mathcal{U}} \mathbb{P}\Big[\tau(x,u) < +\infty \text{ and } X_x^u(t) \in \mathcal{K}\ \forall t \in [0,\tau(x,u)]\Big] = 0.$$

This means that for any control $u \in \mathcal{U}$ and almost every realization $\omega \in \Omega$

$$\tau(x,u)(\omega) = +\infty \quad \text{or} \quad \exists \bar{t} \in [0,\tau(x,u)(\omega)] : X_x^u(\bar{t})(\omega) \notin \mathcal{K}.$$

On the one hand, if $\tau(x,u)(\omega) = +\infty$, $\nexists t$ such that $X_x^u(t)(\omega) \in \mathcal{T}_r$. By assumption (H_f) it follows that

$$f(X_x^u(t),u(t))(\omega) > f_0, \qquad \forall t \geq 0$$

with $f_0 > 0$, that is

$$\exp\Big\{ - \int_0^t f(X_x^u(s),u(s))ds - h(X_x^u(t))\Big\}(\omega) \leq \exp\Big\{ - f_0 t - h(X_x^u(t))\Big\}(\omega) \qquad \forall t \geq 0.$$

On the other hand, if $X_x^u(\bar{t})(\omega) \notin \mathcal{K}$ for a certain $\bar{t} \in [0,\tau(x,u)(\omega)]$, one has $h(X_x^u(\bar{t}))(\omega) = +\infty$. In both cases, for every $u \in \mathcal{U}$ the argument of the expectation in (4.4.1) almost surely has the value 0, implying

$$1 + \mathbb{E}\Big[\sup_{t\geq 0}\Big(- e^{-\int_0^t f(X_x^u(s),u(s))ds - h(X_x^u(t))}\Big)\Big] = 1$$

for every $u \in \mathcal{U}$ from which $v(x) = 1$ follows by the definition of v.

"$\Rightarrow$" We will prove that $\sup_{u\in\mathcal{U}} \mathbb{E}[\inf_{t\geq 0}\ e^{-\int_0^t f(X_x^u(s),u(s))ds - h(X_x^u(t))}] > 0$ for every $x \in \mathcal{D}^{\mathcal{T},\mathcal{K}}$. Let us start observing that, since there exists a control $\nu \in \mathcal{U}$ such that

$$\mathbb{P}\Big[\tau(x,\nu) < +\infty \text{ and } X_x^\nu(t) \in \mathcal{K}\ \forall t \in [0,\tau(x,\nu)]\Big] > 0,$$

then there exist $T, M > 0$ large enough such that for

$$\Omega_1^u := \Big\{\omega \in \Omega : \tau(x,u) < T \text{ and } \max_{t\in[0,\tau(x,u)]} h(X_x^u(t)) \leq M\Big\}$$

one has $\delta := \sup_{u\in\mathcal{U}} \mathbb{P}[\Omega_1^u] > 0$. Indeed, defining

$$\begin{aligned}\Omega_\infty &:= \Big\{\omega \in \Omega : \tau(x,\nu) < +\infty \text{ and } X_x^\nu(t) \in \mathcal{K}\ \forall t \in [0,\tau(x,\nu)]\Big\} \\ &= \Big\{\omega \in \Omega : \tau(x,\nu) < +\infty \text{ and } h(X_x^u(t)) < \infty\ \forall t \in [0,\tau(x,\nu)]\Big\}\end{aligned}$$

and

$$\Omega_n := \Big\{\omega \in \Omega : \tau(x,u) < n \text{ and } \max_{t\in[0,\tau(x,u)]} h(X_x^u(t)) \le n\Big\}$$

one has

$$0 < \mathbb{P}[\Omega_\infty] = \mathbb{P}[\bigcup_{n\ge 0} \Omega_n] \le \sum_{n\ge 0} \mathbb{P}[\Omega_n].$$

Hence, there exists $\bar{n} \in \mathbb{N}$ such that $\mathbb{P}[\Omega_{\bar{n}}] > 0$ and thus $\sup_{u\in\mathcal{U}} \mathbb{P}[\Omega_1^u] > 0$ for $T = M = \bar{n}$.

Moreover, thanks to the assumption of local exponential stabilizability in probability, there exist constants $\lambda, C > 0$ such that for any $y \in \mathcal{T}_r$

$$\sup_{u\in\mathcal{U}} \mathbb{P}[A_y^u] \ge 1 - \frac{\delta}{2}$$

for $A_y^u := \Big\{\omega \in \Omega : \sup_{t\ge 0}\ d_{\mathcal{T}}^+(X_y^u(t))e^{\lambda t} \le C \text{ and } X_y^u(t) \in \mathcal{K}\ \forall t \ge 0\Big\}$.
In what follows we will denote by $\tau = \tau(x,u)$ the hitting time (4.3.1) if no ambiguity arises. For any $u \in \mathcal{U}$ one has (recall that $f \ge 0$):

$$\begin{aligned}
&\mathbb{E}\Big[\inf_{t\ge 0} \exp\Big\{-\int_0^t f(X_x^u(\xi),u(\xi))d\xi - h(X_x^u(t))\Big\}\Big]\\
&\ge \mathbb{E}\Big[\exp\Big\{-\int_0^{+\infty} f(X_x^u(\xi),u(\xi))d\xi - \max_{\xi\in[0,+\infty)} h(X_x^u(\xi))\Big\}\Big]\\
&\ge \int_{\Omega_1^u} \exp\Big\{-\int_0^{+\infty} f(X_x^u(\xi),u(\xi))d\xi - \max_{\xi\in[0,+\infty)} h(X_x^u(\xi))\Big\}d\mathbb{P}\\
&\ge \int_{\Omega_1^u} \exp\Big\{-\int_0^{\tau} f(X_x^u(\xi),u(\xi))d\xi - \int_\tau^{+\infty} f(X_x^u(\xi),u(\xi))d\xi\\
&\quad - \max_{\xi\in[0,\tau]} h(X_x^u(\xi)) \vee \max_{\xi\in[\tau,+\infty)} h(X_x^u(\xi))\Big\}d\mathbb{P}\\
&\ge e^{-M_f T - M} \int_0^T \int_{d_{\mathcal{T}}^+(y)=r} \mathbb{P}\Big[X_\tau^u = dy, \tau = ds, \tau < T, \max_{\xi\in[0,\tau]} h(X_x^u(\xi)) \le M\Big]\\
&\quad \cdot \mathbb{E}\Big[\exp\Big\{-\int_\tau^{+\infty} f(X_x^u(\xi),u(\xi))d\xi - \max_{\xi\in[\tau,+\infty)} h(X_x^u(\xi))\Big\}\Big|_{\substack{X_\tau^u=y,\ \tau=s,\ \tau<T,\\ \max_{\xi\in[0,\tau]} h(X_x^u(\xi))\le M}}\Big]\\
&\ge e^{-M_f T - M} \int_0^T \int_{d_{\mathcal{T}}^+(y)=r} \mathbb{P}\Big[X_\tau^u = dy, \tau = ds, \tau < T, \max_{\xi\in[0,\tau]} h(X_x^u(\xi)) \le M\Big]\\
&\quad \cdot \mathbb{E}\Big[\exp\Big\{-\int_0^{+\infty} f(X_y^{u(s+\cdot)}(\xi),u(s+\xi))d\xi - \max_{\xi\in[0,+\infty)} h(X_y^{u(s+\cdot)}(\xi))\Big\}\Big| X_s^u = y\Big].
\end{aligned}$$

Therefore, applying the Lipschitz continuity of g and h, one has

$$
\begin{aligned}
& e^{-M_f T - M} \sup_{u\in\mathcal{U}} \int_0^T \int_{d^+_{\mathcal{T}}(y)=r} \mathbb{P}\Big[X^u_\tau = dy, \tau = ds, \tau < T, \max_{\xi\in[0,\tau]} h(X^u_x(\xi)) \le M\Big] \\
& \quad \cdot \mathbb{E}\Big[\exp\Big\{ -\int_0^{+\infty} f(X^{u(s+\cdot)}_y(\xi), u(s+\xi))d\xi - \max_{\xi\in[0,+\infty)} h(X^{u(s+\cdot)}_y(\xi))\Big\}\Big| X^u_s = y\Big] \\
& \ge e^{-M_f T - M} \sup_{u\in\mathcal{U}} \int_0^T \int_{d^+_{\mathcal{T}}(y)=r} \mathbb{P}\Big[X^u_\tau = dy, \tau = ds, \tau < T, \max_{\xi\in[0,\tau]} h(X^u_x(\xi)) \le M\Big] \\
& \quad \cdot \mathbb{E}\Big[\chi_{A^u_y} \exp\Big\{ -\int_0^{+\infty} f(X^{u(s+\cdot)}_y(\xi), u(s+\xi))d\xi - \max_{\xi\in[0,+\infty)} h(X^{u(s+\cdot)}_y(\xi))\Big\}\Big| X^u_s = y\Big] \\
& \ge e^{-M_f T - M} \sup_{u\in\mathcal{U}} \int_0^T \int_{d^+_{\mathcal{T}}(y)=r} \mathbb{P}\Big[X^u_\tau = dy, \tau = ds, \tau < T, \max_{\xi\in[0,\tau]} h(X^u_x(\xi)) \le M\Big] \\
& \quad \cdot \mathbb{E}\Big[\chi_{A^u_y} \exp\Big\{ -L_f\int_0^{+\infty} d^+_{\mathcal{T}}(X^{u(s+\cdot)}_y(\xi))d\xi - \max_{\xi\in[0,+\infty)} L d^+_{\mathcal{T}}(X^{u(s+\cdot)}_y(\xi), \mathcal{T})\Big\}\Big| X^u_s = y\Big] \\
& \ge e^{-M_f T - M} \sup_{u\in\mathcal{U}} \int_0^T \int_{d^+_{\mathcal{T}}(y)=r} \mathbb{P}\Big[X^u_\tau = dy, \tau = ds, \tau < T, \max_{\xi\in[0,\tau]} h(X^u_x(\xi)) \le M\Big] \\
& \quad \cdot \mathbb{E}\Big[\chi_{A^u_y} \exp\Big\{ -L_f\int_0^{+\infty} Ce^{-\lambda\xi}d\xi - \max_{\xi\in[0,+\infty)} LCe^{-\lambda\xi}\Big\}\Big| X^u_s = y\Big] \\
& \ge e^{-M_f T}e^{-M}e^{-\frac{CL_f}{\lambda}}e^{-LC} \sup_{u\in\mathcal{U}} \int_0^T \int_{y\in\mathcal{T}_r} \mathbb{E}\Big[\chi_{A^u_y}\Big| X^u_s = y\Big] \\
& \quad \cdot \mathbb{P}\Big[X^u_\tau = dy, \tau = ds, \tau(x,u) < T, \max_{\xi\in[0,\tau]} h(X^u_x(\xi)) \le M\Big] \\
& = e^{-M_f T}e^{-M}e^{-\frac{CL_f}{\lambda}}e^{-LC} \sup_{u\in\mathcal{U}} \mathbb{P}\Big[\Omega^u_1 \cap A^u_{X^u_\tau}\Big] > 0
\end{aligned}
$$

where for the last inequality we used the fact that (thanks again to the arguments in Remark 4.3.1) one has $\sup_{u\in\mathcal{U}} \mathbb{P}[\Omega^u_1 \cap A^u_{X^u_\tau}] > 0$. □

Remark 4.4.2. The definition of the function v is based on a similar construction used in [104] for a deterministic controlled setting. That paper shows that in the deterministic setting the domain of controllability can alternatively be characterized by a second function, whose definition, translated to the stochastic framework, would be

$$
V(x) = \inf_{u\in\mathcal{U}} \mathbb{E}\Big[\sup_{t\ge 0}\int_0^t f(X^u_x(s), u(s))ds + h(X^u_x(t))\Big]. \tag{4.4.2}
$$

A little computation using Jensen's inequality shows the relation

$$
\Big\{x \in \mathbb{R}^d : V(x) < +\infty\Big\} \subseteq \Big\{x \in \mathbb{R}^d : v(x) < 1\Big\}.
$$

Since, however, it is not clear whether the opposite inclusion holds in the stochastic setting, we will exclusively work with v in the remainder of this chapter.

4.5 The PDE characterization of $\mathcal{D}^{\mathcal{T},\mathcal{K}}$

After having shown that $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ can be expressed as a sub-level set of v, we now proceed to the second main result of the chapter, the PDE characterization of v and thus of

$\mathcal{D}^{\mathcal{T},\mathcal{K}}$. In order to derive the PDE which is solves by v, we need to establish a dynamic programming principle for v. However, as already pointed out in Chapter 3 Section 3.5, the presence of the supremum inside the expectation in the definition of v prohibits the direct use of the standard dynamic programming techniques. In fact, exactly as for the value function associated with the cost (3.4.1) in Section 3.5, v does not satisfy a fundamental concatenation property that is usually the main tool necessary for the derivation of the associated partial differential equation. To avoid this difficulty, we follow the classical approach, exploited also in Section 3.5, to reformulate the problem by adding the auxiliary variable $y \in \mathbb{R}$ that keeps track of the running maximum. For this reason we introduce the function $\vartheta : \mathbb{R}^d \times [-1,0] \to [0,1]$ defined as follows:

$$\vartheta(x,y) := \inf_{u\in\mathcal{U}} \left\{ 1 + \mathbb{E}\left[\sup_{t\geq 0} \left(-e^{-\int_0^t f(X_x^u(s),u(s))ds - h(X_x^u(t))} \right) \vee y \right] \right\}. \tag{4.5.1}$$

We point out that

$$\vartheta(x,-1) = v(x) \qquad \forall x \in \mathbb{R}^d,$$

therefore ϑ can still be used for characterizing the set $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ and one has

$$\mathcal{D}^{\mathcal{T},\mathcal{K}} = \left\{ x \in \mathbb{R}^d : \vartheta(x,-1) < 1 \right\}. \tag{4.5.2}$$

Furthermore, it follows from Theorem 4.4.1 that

$$\vartheta(x,y) = \begin{cases} 1+y & \text{on} \quad \mathcal{T} \times [-1,0] \\ 1 & \text{on} \quad (\mathcal{D}^{\mathcal{T},\mathcal{K}})^C \times [-1,0]. \end{cases} \tag{4.5.3}$$

In what follows we will also denote

$$F(t,x,u) := \int_0^t f(X_x^u(s), u(s))ds,$$

so that using this notation the function ϑ reads

$$\vartheta(x,y) = \inf_{u\in\mathcal{U}} \left\{ 1 + \mathbb{E}\left[\sup_{t\geq 0} \left(-e^{-F(t,x,u) - h(X_x^u(t))} \right) \vee y \right] \right\}.$$

For the new state variable y we can define the following "maximum dynamics":

$$Y_{x,y}^u(\cdot) := e^{F(\cdot,x,u)} \left(y \vee \sup_{t\in[0,\cdot\,]} \left(-e^{-F(t,x,u) - h(X_x^u(t))} \right) \right) \tag{4.5.4}$$

We remark that $Y_{x,y}^u(t) \in [-1,0]$ for any $u \in \mathcal{U}$, $t \geq 0$ and $(x,y) \in \mathbb{R}^d \times [-1,0]$. Moreover one has

$$1 + y \leq \vartheta(x,y) \leq 1, \qquad \forall x \in \mathbb{R}^d, y \in [-1,0]. \tag{4.5.5}$$

We are now able to prove a DPP for the function ϑ. Since no information is available at the moment on the regularity of ϑ, we state the weak version of the DPP presented in [63] involving the semi-continuous envelopes of ϑ. Let us denote by ϑ^* and ϑ_* respectively the upper and lower semi-continuous envelope of ϑ. One has:

Lemma 4.5.1. *Let assumptions* (H_b),(H_σ),(H_f) *and* (H_h) *be satisfied. Then for any finite stopping time* $\theta \geq 0$ *measurable with respect to the filtration, one has*

$$\inf_{u\in\mathcal{U}} \mathbb{E}\Big[e^{-F(\theta,x,u)}\vartheta_*(X_x^u(\theta), Y_{x,y}^u(\theta))) + \int_0^\theta f(X_x(s), u(s))e^{-F(s,x,u)}ds\Big]$$
$$\leq \vartheta(x,y) \leq \inf_{u\in\mathcal{U}} \mathbb{E}\Big[e^{-F(\theta,x,u)}\vartheta^*(X_x^u(\theta), Y_{x,y}^u(\theta))) + \int_0^\theta f(X_x^u(s), u(s))e^{-F(s,x,u)}ds\Big].$$

For a rigorous proof of this result we refer to [63]. Here, we only show the main steps that lead to our formulation of the DPP in the non-controlled and continuous case.

Sketch of the proof of Lemma 4.5.1. For any finite stopping time $\theta \geq 0$ one has

$$\begin{aligned}
&\vartheta(x,y) - 1\\
&= \mathbb{E}\Big[\sup_{t\geq 0}\big(-e^{-F(t,x)-h(X_x(t))}\big) \vee y\Big]\\
&= \mathbb{E}\Big[\sup_{t\geq \theta}\big(-e^{-F(t,x)-h(X_x(t))}\big) \vee \sup_{t\in[0,\theta]}\big(-e^{-F(t,x)-h(X_x(t))}\big) \vee y\Big]\\
&= \mathbb{E}\Big[e^{-F(\theta,x)}\sup_{t\geq \theta}\big(-e^{-\int_\theta^t f(X_x(s))ds-h(X_x(t))}\big) \vee \sup_{t\in[0,\theta]}\big(-e^{-F(t,x)-h(X_x(t))}\big) \vee y\Big]\\
&= \mathbb{E}\Big[e^{-F(\theta,x)}\Big\{\sup_{t\geq \theta}\big(-e^{-\int_\theta^t f(X_x(s))ds-h(X_x(t))}\big) \vee Y_{x,y}(\theta)\Big\}\Big]
\end{aligned}$$

where the property of the maximum $(a\cdot b)\vee c = a\cdot(b \vee \frac{c}{a})$, $\forall a,b,c \in \mathbb{R}, a > 0$, is used. Applying now the tower property of the expectation one obtains

$$\begin{aligned}
&\vartheta(x,y)\\
&= 1 + \mathbb{E}\Big[\mathbb{E}\Big[e^{-F(\theta,x)}\Big\{\sup_{t\geq 0}\big(-e^{-F(t,X_x(\theta))-h(X_{X_x(\theta)}(t))}\big) \vee Y_{x,y}(\theta)\Big\}\ \Big|\ \mathbb{F}_\theta\Big]\ \Big]\\
&= 1 + \mathbb{E}\Big[e^{-F(\theta,x)}\mathbb{E}\Big[\sup_{t\geq 0}\big(-e^{-F(t,X_x(\theta))-h(X_{X_x(\theta)}(t))}\big) \vee Y_{x,y}(\theta)\ \Big|\ \mathbb{F}_\theta\Big]\ \Big]\\
&= 1 + \mathbb{E}\Big[e^{-F(\theta,x)}\Big(\vartheta(X_x(\theta), Y_{x,y}(\theta)) - 1\Big)\Big]
\end{aligned}$$

and the result just follows observing that $1 - e^{-F(\theta,x)} = \int_0^\theta f(X_x(s))e^{-F(s,x)}ds$. $\square$

Using the DPP from Lemma 4.5.1, we can now show that ϑ is actually continuous.

Proposition 4.5.2. *Let assumptions* (H_b),(H_σ),$(H'_{\mathcal{T}})$,$(H'_{\mathcal{K}})$,(H_f) *and* (H_h) *be satisfied. Then the function* ϑ *from* (4.5.1) *is continuous in* $\mathbb{R}^{d+1}$.

Proof. The continuity with respect to y is trivial and one has

$$|\vartheta(x,y) - \vartheta(x,y')| \leq |y - y'|.$$

For what concerns the continuity with respect to x, in $(\mathcal{D}^{\mathcal{T},\mathcal{K}})^C$ and $\mathcal{T}$ there is nothing to prove thanks to (4.5.3).

We start by proving the continuity at the boundary of $\mathcal{T}$. Let $x_0 \in \partial\mathcal{T}$. We aim to prove that for any ε there exists $\delta > 0$ such that for $x \in B(x_0,\delta)$ one has

$$\vartheta(x,y) - \vartheta(x_0,y) = \vartheta(x,y) - (1+y) \leq \varepsilon. \tag{4.5.6}$$

For $\delta > 0$ small enough we can assume that $B(x_0, \delta) \subset \mathcal{T}_r$. Hence, for this choice of δ there exists $\lambda > 0$ such that for any $\varepsilon > 0$ there exists a constant C_ε and a control ν such that one has

$$\mathbb{P}[A_x^C] \leq \frac{\varepsilon}{2}$$

for $A_x := \left\{ \omega \in \Omega : \sup_{t \geq 0} d_{\mathcal{T}}^+(X_x^\nu(t)) e^{\lambda t} \leq C_\varepsilon d_{\mathcal{T}}^+(x) \text{ and } X_x^\nu(t) \in \mathcal{K}\ \forall t \geq 0 \right\}$.
From the definition of ϑ and the monotonicity of the exponential one has

$$\begin{aligned}
&\vartheta(x,y) - (1+y) \\
&= \vartheta(x,y) - \big(1 + (-1) \vee y\big) \\
&\leq \mathbb{E}\Big[\sup_{t \geq 0} (-e^{-F(t,x,\nu) - h(X_x^\nu(t))}) \vee y - \big((-1) \vee y\big)\Big] \\
&\leq \mathbb{E}\Big[1 + \sup_{t \geq 0}(-e^{-F(t,x,\nu) - h(X_x^\nu(t))})\Big] \\
&= \mathbb{E}\Big[1 - \exp\Big\{ - \sup_{t \geq 0}(F(t,x,\nu) + h(X_x^\nu(t)))\Big\}\Big] \\
&= \int_{A_x} \Big(1 - \exp\Big\{ - \sup_{t \geq 0}(F(t,x,\nu) + h(X_x^\nu(t)))\Big\}\Big) d\mathbb{P} \\
&\quad + \int_{A_x^C} \Big(1 - \exp\Big\{ - \sup_{t \geq 0}(F(t,x,\nu) + h(X_x^\nu(t)))\Big\}\Big) d\mathbb{P} \\
&\leq \int_{A_x} \Big(1 - \exp\Big\{ - \sup_{t \geq 0}(F(t,x,\nu) + h(X_x^\nu(t)))\Big\}\Big) d\mathbb{P} + \frac{\varepsilon}{2}
\end{aligned}$$

for every $T > 0$. Therefore in order to conclude (4.5.6) it will be sufficient to estimate the integral taking into account the events in A_x.

For sufficiently small $\delta > 0$ we obtain $C_\varepsilon d_{\mathcal{T}}^+(x) < r$ and thus $X_x^\nu(t,\omega) \in \mathcal{T}_r$ for all $\omega \in A_x$, all $t \geq 0$ and all $x \in B(x_0, \delta)$. Thus, since $\mathcal{T}_r$ is a compact subset of C, the function h is Lipschitz with constant L along all these trajectories. Since f is Lipschitz, too, and since $f(\xi, u) = h(\xi) = 0\ \forall \xi \in \mathcal{T}, u \in U$, for any $t \geq 0$ one has

$$f(X_x^\nu(t), \nu(t)) \leq L_f d_{\mathcal{T}}^+(X_x^\nu(t)) \quad \text{and} \quad h(X_x^\nu(t)) \leq L d_{\mathcal{T}}^+(X_x^\nu(t)).$$

Using these inequalities and the definition of A_x, we obtain

$$\begin{aligned}
&\int_{A_x} \Big(1 - \exp\Big\{ - \sup_{t \geq 0}(F(t,x,\nu) + h(X_x^\nu(t)))\Big\}\Big) d\mathbb{P} \\
&\leq \int_{A_x} \Big(1 - \exp\Big\{ - \int_0^{+\infty} f(X_x^\nu(t), \nu(t)) dt - \sup_{t \geq 0} h(X_x^\nu(t)))\Big\}\Big) d\mathbb{P} \\
&\leq \int_{A_x} \Big(1 - \exp\Big\{ - \int_0^{+\infty} L_f d_{\mathcal{T}}^+(X_x^\nu(t)) dt - \sup_{t \geq 0} L d_{\mathcal{T}}^+(X_x^\nu(t))\Big\}\Big) d\mathbb{P} \\
&\leq \int_{A_x} \Big(1 - \exp\Big\{ - \int_0^{+\infty} L_f C_\varepsilon d_{\mathcal{T}}^+(x) e^{-\lambda t} dt - \sup_{t \geq 0} L C_\varepsilon d_{\mathcal{T}}^+(x) e^{-\lambda t}\Big\}\Big) d\mathbb{P} \\
&\leq \int_{A_x} 1 - e^{-(L_f/\lambda + L) C_\varepsilon \delta} d\mathbb{P} \ \leq \ 1 - e^{-(L_f/\lambda + L) C_\varepsilon \delta}.
\end{aligned}$$

Now, choosing $\delta > 0$ such that

$$\left(\frac{L_f}{\lambda} + L\right) C_\varepsilon \delta \leq -\ln(1 - \varepsilon/2)$$

, we have

$$1 - e^{-(L_f/\lambda + L)C_\varepsilon \delta} \leq \varepsilon/2$$

and thus

$$\vartheta(x,y) - (1+y) \leq \int_{A_x} \left(1 - \exp\left\{ - \sup_{t \geq 0}(F(t,x,\nu) + h(X_x^\nu(t)))\right\}\right) d\mathbb{P} + \frac{\varepsilon}{2} \leq \varepsilon,$$

for any x with $d_{\mathcal{T}}^+(x) < \delta$, which proves (4.5.6) and thus continuity at $\partial \mathcal{T}$.

The proof of the theorem is concluded proving the continuity in $\mathbb{R}^d \setminus \mathcal{T}$. We point out that we already know that $\vartheta(x,y) = 1 + y$ in $(\mathcal{D}^{\mathcal{T},\mathcal{K}})^C$, however the proof that follows is independent of whether $x \in \mathcal{D}^{\mathcal{T},\mathcal{K}}$ or not. Let $x \in \overline{\mathcal{D}^{\mathcal{T},\mathcal{K}}} \setminus \mathcal{T}$ and $\xi \in B(x,\delta)$. From the DPP (Lemma 4.5.1), for any $y \in [-1,0]$ and any finite stopping time θ, there exists a control $\nu = \nu_\varepsilon \in \mathcal{U}$ such that

$$\begin{aligned}\vartheta(\xi,y) - \vartheta(x,y) \leq \mathbb{E}\Big[& e^{-F(\theta,\xi,\nu)} \vartheta^*(X_\xi^\nu(\theta), Y_{\xi,y}^\nu(\theta)) - e^{-F(\theta,\xi,\nu)} \\ & - e^{-F(\theta,x,\nu)} \vartheta_*(X_x^\nu(\theta), Y_{x,y}^\nu(\theta)) + e^{-F(\theta,x,\nu)}\Big] + \frac{\varepsilon}{4}.\end{aligned}$$

In order to prove the result we will use the continuity at $\mathcal{T}$ we proved above. We can in fact state that for any $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that

$$\vartheta^*(z,y) \leq 1 + y + \frac{\varepsilon}{4} \quad \text{if} \quad d_{\mathcal{T}}^+(z) \leq \eta_\varepsilon.$$

Let $T \geq -\frac{\ln(\varepsilon/4)}{f^*}$ and $0 < R \leq \frac{\varepsilon/4}{L_h + L_f T}$ where $f^* := \inf_{\{x : d_{\mathcal{T}}^+(x) \geq \eta_\varepsilon/2\}} f(x,\nu) > 0$ and L_h, L_f are, respectively, the Lipschitz constant of $e^{-h(x)}$ and f. Denoting

$$E := \left\{ \omega \in \Omega : \sup_{t \in [0,T]} |X_x^\nu(t) - X_\xi^\nu(t)| \geq R \right\},$$

under assumptions (H_b) and (H_σ) we can choose δ sufficiently small such that $\mathbb{P}[E] \leq \frac{\varepsilon}{8}$. Then (recalling that $\vartheta^*, \vartheta_* \in [0,1]$), we have

$$\begin{aligned}&\int_E \Big(e^{-F(\theta,\xi,\nu)} \vartheta^*(X_\xi^\nu(\theta), Y_{\xi,y}^\nu(\theta)) - e^{-F(\theta,\xi,\nu)} \\ &\quad - e^{-F(\theta,x,\nu)} \vartheta_*(X_x^\nu(\theta), Y_{x,y}^\nu(\theta)) + e^{-F(\theta,x,\nu)}\Big) d\mathbb{P} \\ &\leq \int_E \left(e^{-F(\theta,x,\nu)} + e^{-F(\theta,\xi,\nu)}\right) d\mathbb{P} \leq 2\mathbb{P}[E] \leq \frac{\varepsilon}{4}.\end{aligned} \tag{4.5.7}$$

Let us now define the stopping time

$$\bar{\theta} := \inf\left\{ t \geq 0 : d_{\mathcal{T}}^+(X_x^\nu(t)) \leq \eta_\varepsilon \right\}$$

with the convention that $\bar\theta(\omega) = T$ if $d^+_{\mathcal{T}}(X^\nu_x(t)(\omega)) > \eta_\varepsilon, \forall t \in [0,T]$ (this ensures the finiteness of the stopping time needed for the DPP). Thanks to (4.5.7) (which holds for an arbitrary stopping time), we can write

$$\mathbb{E}\Big[e^{-F(\bar\theta,\xi,\nu)}\vartheta^*(X^\nu_\xi(\bar\theta), Y^\nu_{\xi,y}(\bar\theta)) - e^{-F(\bar\theta,\xi,\nu)} - e^{-F(\bar\theta,x,\nu)}\vartheta_*(X^\nu_x(\bar\theta), Y^\nu_{x,y}(\bar\theta)) + e^{-F(\bar\theta,x,\nu)}\Big]$$
$$\leq \frac{\varepsilon}{4} + \int_{E^C} \cdots = \frac{\varepsilon}{4} + \int_{E^C\cap\{\bar\theta<T\}} \cdots + \int_{E^C\cap\{\bar\theta=T\}} \cdots$$

and we will provide estimates separately for the last two integrals.
In $E^C \cap \{\bar\theta = T\}$, using again $\vartheta_*, \vartheta^* \in [0,1]$, we get

$$\begin{aligned}
&\int_{E^C\cap\{\bar\theta=T\}} \Big(e^{-F(T,\xi,\nu)}\vartheta^*(X^\nu_\xi(T), Y^\nu_{\xi,y}(T)) - e^{-F(T,\xi,\nu)} \\
&\qquad\qquad - e^{-F(T,x,\nu)}\vartheta_*(X^\nu_x(T), Y^\nu_{x,y}(T)) + e^{-F(T,x,\nu)}\Big) d\mathbb{P} \\
&\leq \int_{E^C\cap\{\bar\theta=T\}} e^{-F(T,x,\nu)} d\mathbb{P} \leq e^{-f^*T} \leq \frac{\varepsilon}{4}
\end{aligned}$$

thanks to the choice of T.

In $E^C \cap \{\bar\theta < T\}$ we have

$$\begin{aligned}
&\int_{E^C\cap\{\bar\theta<T\}} \Big(e^{-F(\bar\theta,\xi,\nu)}\vartheta^*(X^\nu_\xi(\bar\theta), Y^\nu_{\xi,y}(\bar\theta)) - e^{-F(\bar\theta,\xi,\nu)} \\
&\quad - e^{-F(\bar\theta,x,\nu)}\vartheta_*(X^\nu_x(\bar\theta), Y^\nu_{x,y}(\bar\theta)) + e^{-F(\bar\theta,x,\nu)}\Big) d\mathbb{P} \\
&\leq \int_{E^C\cap\{\bar\theta<T\}} \Big(e^{-F(\bar\theta,\xi,\nu)}\big(1 + Y^\nu_{\xi,y}(\bar\theta) + \frac{\varepsilon}{4}\big) - e^{-F(\bar\theta,\xi,\nu)} \\
&\quad - e^{-F(\bar\theta,x,\nu)}\big(1 + Y^\nu_{x,y}(\bar\theta)\big) + e^{-F(\bar\theta,x,\nu)}\Big) d\mathbb{P} \\
&= \int_{E^C\cap\{\bar\theta<T\}} \Big(e^{-F(\bar\theta,\xi,\nu)}Y^\nu_{\xi,y}(\bar\theta) - e^{-F(\bar\theta,x,\nu)}Y^\nu_{x,y}(\bar\theta)\Big) d\mathbb{P} + \frac{\varepsilon}{4}
\end{aligned}$$

where we used the fact that, in virtue of (4.5.5), $\vartheta_*(x,y) \geq 1+y$. Recalling the definition of the variable $Y(\cdot)$ given by (4.5.4) and because of assumptions (H_f) and (H_h), we have

$$\begin{aligned}
&\int_{E^C\cap\{\bar\theta<T\}} \left(e^{-F(\bar\theta,\xi,\nu)} Y^\nu_{\xi,y}(\bar\theta) - e^{-F(\bar\theta,x,\nu)} Y^\nu_{x,y}(\bar\theta)\right) d\mathbb{P} \\
&= \int_{E^C\cap\{\bar\theta<T\}} \left(\sup_{t\in[0,\bar\theta]} (-e^{-F(t,\xi,\nu)-h(X^\nu_\xi(t))}) \vee y - \sup_{t\in[0,\bar\theta]} (-e^{-F(t,x,\nu)-h(X^\nu_x(t))}) \vee y\right) d\mathbb{P} \\
&\le \int_{E^C\cap\{\bar\theta<T\}} \left(\sup_{t\in[0,\bar\theta]} |e^{-F(t,\xi,\nu)-h(X^\nu_\xi(t))} - e^{-F(t,x,\nu)-h(X^\nu_x(t))}|\right) d\mathbb{P} \\
&\le \int_{E^C\cap\{\bar\theta<T\}} \left(\sup_{t\in[0,\bar\theta]} e^{-F(t,\xi,\nu)} |e^{-h(X^\nu_\xi(t))} - e^{-h(X^\nu_x(t))}|\right. \\
&\qquad\qquad \left. + \sup_{t\in[0,\bar\theta]} e^{-h(X^\nu_\xi(t))} |e^{-F(t,\xi,\nu)} - e^{-F(t,x,\nu)}|\right) d\mathbb{P} \\
&\le \int_{E^C\cap\{\bar\theta<T\}} \left(\sup_{t\in[0,\bar\theta]} |e^{-h(X^\nu_\xi(t))} - e^{-h(X^\nu_x(t))}| + \sup_{t\in[0,\tau]} |e^{-F(t,\xi,\nu)} - e^{-F(t,x,\nu)}|\right) d\mathbb{P} \\
&\le (L_h + L_f T) \int_{E^C\cap\{\bar\theta<T\}} \left(\sup_{t\in[0,T]} |X^\nu_\xi(t) - X^\nu_x(t)|\right) d\mathbb{P} \le \frac{\varepsilon}{4}
\end{aligned}$$

thanks to the choice of R. □

Thanks to Lemma 4.5.1 and the continuity of ϑ, we can finally characterize ϑ as a solution, in the viscosity sense, of a second order Hamilton-Jacobi-Bellman equation. To this end, we define the open domain $\mathcal{O} \subset \mathbb{R}^d \times [-1, 0]$ by

$$\mathcal{O} = \left\{(x, y) \in \mathbb{R}^{d+1} : -e^{-h(x)} < y < 0\right\}$$

and the following two components of its boundary

$$\partial_1\mathcal{O} := \left\{(x, y) \in \overline{\mathcal{O}} : y = 0\right\}$$

$$\partial_2\mathcal{O} := \left\{(x, y) \in \overline{\mathcal{O}} : y = -e^{-h(x)},\ y < 0\right\}.$$

Remark 4.5.3. We point out that thanks to the relation

$$\vartheta(x, y) = \vartheta(x, -e^{-h(x)}) \qquad \forall y \le -e^{-h(x)}$$

it is sufficient to determine the values of ϑ in $\overline{\mathcal{O}}$ in order to characterize ϑ in the whole domain of definition $\mathbb{R}^d \times [-1, 0]$. We also remark that $\vartheta(x, 0) = 1$ for any $x \in \mathbb{R}^d$. Moreover if the assumption of boundedness of $\mathcal{K}$ holds, $\mathcal{O}$ is bounded too.

Let us consider the following Hamiltonian $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathcal{S}^d \to \mathbb{R}$,

$$H(x, y, r, p, q, Q) := \sup_{u\in U} \left\{f(x, u)(r - 1) - p \cdot b(x, u) - \frac{1}{2} Tr[\sigma\sigma^T(x, u)Q] - q\, f(x, u)y\right\}. \tag{4.5.8}$$

The following theorem holds.

Theorem 4.5.4. *Let assumptions (H_b),(H_σ),$(H'_\mathcal{T})$,$(H'_\mathcal{K})$,(H_f) and (H_h) be satisfied. Then ϑ is a continuous viscosity solution of*

$$\begin{cases} H(x, y, \vartheta, D_x\vartheta, \partial_y\vartheta, D^2_x\vartheta) = 0 & \text{in } \mathcal{O} \\ \vartheta = 1 & \text{on } \partial_1\mathcal{O} \\ -\partial_y\vartheta = 0 & \text{on } \partial_2\mathcal{O}. \end{cases} \tag{4.5.9}$$

We refer to [84, Definition 7.4] (see also Definition 3.5.3 in Chapter 3) for the definition of a viscosity solution for equation (4.5.9). It is in fact well-known that boundary conditions may have to be considered in a weak sense in order to obtain existence of a solution. It means that for the viscosity sub-solution (resp. super-solution) of equation (4.5.9), we will ask that on the boundary $\partial_2\mathcal{O}$ the inequality

$$\min\left(H(x,y,\vartheta,D_x\vartheta,\partial_y\vartheta,D_x^2\vartheta),-\partial_y\vartheta\right)\le 0$$

$$\left(\text{resp.}\quad \max\left(H(x,y,\vartheta,D_x\vartheta,\partial_y\vartheta,D_x^2\vartheta),-\partial_y\vartheta\right)\ge 0\right)$$

holds in the viscosity sense. In contrast, the condition on $\partial_1\mathcal{O}$ is assumed in the strong sense.
The proof that follows makes use of the same arguments used in the proof of Theorem 3.5.2 in Chapter 3.

Proof of Theorem 4.5.4. The boundary condition on $\partial_1\mathcal{O}$ follows directly by the definition of ϑ. Let us start proving the sub-solution property.

Let be $\varphi\in C^{2,1}(\overline{\mathcal{O}})$ such that $\vartheta-\varphi$ attains a maximum at point $(\bar{x},\bar{y})\in\overline{\mathcal{O}}$ and let us assume $\bar{y}<0$. We need to show

$$H(\bar{x},\bar{y},\vartheta(\bar{x},\bar{y}),D_x\varphi(\bar{x},\bar{y}),\partial_y\varphi(\bar{x},\bar{y}),D_x^2\varphi(\bar{x},\bar{y}))\le 0 \tag{4.5.10}$$

if $(\bar{x},\bar{y})\notin\partial_2\mathcal{O}$ and

$$\min\left(H(\bar{x},\bar{y},\vartheta(\bar{x},\bar{y}),D_x\varphi(\bar{x},\bar{y}),\partial_y\varphi(\bar{x},\bar{y}),D_x^2\varphi(\bar{x},\bar{y})),-\partial_y\varphi(\bar{x},\bar{y})\right)\le 0 \tag{4.5.11}$$

if $(\bar{x},\bar{y})\in\partial_2\mathcal{O}$.

Without loss of generality we can always assume that $(\bar{x},\bar{y})$ is a strict local maximum point (let us say in a ball of radius r) and that $\vartheta(\bar{x},\bar{y})=\varphi(\bar{x},\bar{y})$. Using continuity arguments, for any $u\in\mathcal{U}$ and for almost every $\omega\in\Omega$ we can find $\theta:=\theta^u$ small enough such that

$$(X^u_{\bar{x}}(\theta)\ ,\ Y^u_{\bar{x},\bar{y}}(\theta))(\omega)\in B((\bar{x},\bar{y}),r).$$

Let us in particular consider a constant control $u(t)\equiv u\in U$. Thanks to Lemma 4.5.1 one has

$$\varphi(\bar{x},\bar{y})\le\mathbb{E}\Big[e^{-F(\theta,\bar{x},u)}\varphi(X^u_{\bar{x}}(\theta),Y^u_{\bar{x},\bar{y}}(\theta))+\int_0^\theta f(X^u_{\bar{x}}(s),u)e^{-F(s,\bar{x},u)}ds\Big]. \tag{4.5.12}$$

We now take into account two different cases, depending on whether or not we are in $\partial_2\mathcal{O}$.

— Case 1: $\bar{y}>-e^{-h(\bar{x})}$. In this case (since we are inside $\mathcal{O}$) for almost every $\omega\in\Omega$, taking the stopping time $\theta(\omega)$ small enough, we can say

$$e^{F(\theta,\bar{x},u)}(\bar{y}\vee\sup_{t\in[0,\theta]}(-e^{-F(t,\bar{x},u)-h(X^u_{\bar{x}}(t))}))(\omega)=(e^{F(\theta,\bar{x},u)}\bar{y})(\omega).$$

Therefore from (4.5.12), for this choice of the stopping time θ, for any $u\in U$ we obtain

$$\mathbb{E}\Big[\varphi(\bar{x},\bar{y})-e^{-F(\theta,\bar{x},u)}\varphi(X^u_{\bar{x}}(\theta),e^{F(\theta,\bar{x},u)}\bar{y})+\int_0^\theta f(X^u_{\bar{x}}(s),u)e^{-F(s,\bar{x},u)}ds\Big]\le 0 \tag{4.5.13}$$

which yields

$$\mathbb{E}\Big[\int_0^\theta -d\Big(e^{-F(s,\bar{x},u)}\varphi(X_{\bar{x}}^u(s), e^{F(s,\bar{x},u)}\bar{y})\Big) + f(X_{\bar{x}}^u(s),u)e^{-F(s,\bar{x},u)}ds\Big] \leq 0.$$

Applying the Ito's formula we have

$$\begin{aligned}
&d\Big(e^{-F(s,\bar{x},u)}\varphi(X_{\bar{x}}^u(s), e^{F(s,\bar{x},u)}\bar{y})\Big)\\
&= e^{-F(s,\bar{x},u)}\Big\{ - f(X_{\bar{x}}^u(s),u)\varphi(X_{\bar{x}}^u(s), e^{F(s,\bar{x},u)}\bar{y})\\
&\quad + D_x\varphi(X_{\bar{x}}^u(s), e^{F(s,\bar{x},u)}\bar{y})\cdot dX_{\bar{x}}^u(s) + \partial_y\varphi(X_{\bar{x}}^u(s), e^{F(s,\bar{x},u)}\bar{y})\ f(X_{\bar{x}}^u(s),u)\bar{y}\\
&\quad + \frac{1}{2}Tr[\sigma\sigma^T(X_{\bar{x}}^u(s),u)D_x^2\varphi(X_{\bar{x}}^u(s), e^{F(s,\bar{x},u)}\bar{y})]\Big\}.
\end{aligned}$$

Then, replacing the stopping time θ by $\theta_h := \theta \wedge h$ we get

$$\begin{aligned}
&\mathbb{E}\Big[\frac{1}{h}\int_0^{\theta_h} e^{-F(s,\bar{x},u)}\Big\{ - f(X_{\bar{x}}^u(s),u)\varphi(X_{\bar{x}}^u(s), e^{F(s,\bar{x},u)}\bar{y})\\
&+ D_x\varphi(X_{\bar{x}}^u(s), e^{F(s,\bar{x},u)}\bar{y})\cdot b(X_{\bar{x}}^u(s),u) + \partial_y\varphi(X_{\bar{x}}^u(s), e^{F(s,\bar{x},u)}\bar{y})\ f(X_{\bar{x}}^u(s),u)\bar{y}\\
&\quad + \frac{1}{2}Tr[\sigma\sigma^T(X_{\bar{x}}^u(s),u)D_x^2\varphi(X_{\bar{x}}^u(s), e^{F(s,\bar{x},u)}\bar{y})]\Big\}ds\Big] \leq 0.
\end{aligned}$$

Letting $h \to 0$ and observing that for ω fixed $\theta_h{=}h$ holds for $h>0$ sufficiently small, we can apply the mean value theorem inside the integral for any fixed ω. In this way, applying also the dominated convergence theorem, we finally obtain at $(\bar{x},\bar{y})$

$$f(\bar{x},u)(\varphi-1) - D_x\varphi\cdot b(\bar{x},u) - \frac{1}{2}Tr[\sigma\sigma^T(\bar{x},u)D_x^2\varphi] - \partial_y\varphi f(\bar{x},u)\bar{y} \leq 0 \quad \forall u\in U$$

and then thanks to the arbitrariness of u

$$H(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}),D_x\varphi(\bar{x},\bar{y}),\partial_y\varphi(\bar{x},\bar{y}),D_x^2\varphi(\bar{x},\bar{y})) \leq 0,$$

i.e., (4.5.10).

— Case 2: $\bar{y} = -e^{-h(\bar{x})}$. If $-\partial_y\varphi(\bar{x},\bar{y}) \leq 0$, then (4.5.11) holds. Hence, let us assume that

$$-\partial_y\varphi(\bar{x},\bar{y}) > 0.$$

This means that in a neighborhood of $(\bar{x},\bar{y})$

$$\varphi(x,y_1) \geq \varphi(x,y_2) \qquad \text{if } y_1 \leq y_2.$$

For almost every $\omega\in\Omega$ and for $\theta(\omega)$ small enough, the point

$$\Big(X_{\bar{x}}^u(\theta)\ ,\ e^{F(\theta,\bar{x},u)}(\bar{y}\vee \sup_{t\in[0,\theta]}(-e^{-F(t,\bar{x},u)-h(X_{\bar{x}}^u(t))}))\Big)$$

is in this neighborhood. Because of

$$Y_{\bar{x},\bar{y}}^u(\theta) = e^{F(\theta,\bar{x},u)}\Big(\bar{y}\vee \sup_{t\in[0,\theta]}(-e^{-F(t,\bar{x},u)-h(X_{\bar{x}}^u(t))})\Big) \geq e^{F(\theta,\bar{x},u)}\bar{y}$$

we obtain for any $u \in U$

$$
\begin{aligned}
&\varphi(\bar{x}, \bar{y}) \\
&\leq \mathbb{E}\Big[e^{-F(\theta,\bar{x},u)}\Big\{\varphi(X^u_{\bar{x}}(\theta), Y^u_{\bar{x},\bar{y}}(\theta)) + \int_0^\theta f(X^u_{\bar{x}}(s), u)e^{-F(s,\bar{x},u)}ds\Big\}\Big] \\
&\leq \mathbb{E}\Big[e^{-F(\theta,\bar{x},u)}\Big\{\varphi(X^u_{\bar{x}}(\theta), e^{F(\theta,\bar{x},u)}\bar{y}) + \int_0^\theta f(X^u_{\bar{x}}(s), u)e^{-F(s,\bar{x},u)}ds\Big\}\Big]
\end{aligned}
$$

from which we have again (4.5.13) and thus

$$H(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}), D_x\varphi(\bar{x}, \bar{y}), \partial_y\varphi(\bar{x}, \bar{y}), D^2_x\varphi(\bar{x}, \bar{y})) \leq 0,$$

implying (4.5.11).

For proving the super-solution property let us assume that $\vartheta - \varphi$ attains a strict maximum in $(\bar{x}, \bar{y})$. Starting again from the DPP and taking the stopping time θ small enough one has

$$\varphi(\bar{x}, \bar{y}) \geq \inf_{u\in\mathcal{U}} \mathbb{E}\Big[e^{-F(\theta,\bar{x},u)}\varphi(X^u_{\bar{x}}(\theta), Y^u_{\bar{x},\bar{y}}(\theta)) + \int_0^\theta f(X^u_{\bar{x}}(s), u)e^{-F(s,\bar{x},u)}ds\Big].$$

If either $\bar{y} > -e^{-h(\bar{x})}$ or $\bar{y} = -e^{-h(\bar{x})}$ and $-\partial_y\varphi(\bar{x}, \bar{y}) < 0$ we get, for θ small enough

$$\varphi(\bar{x}, \bar{y}) \geq \inf_{u\in\mathcal{U}} \mathbb{E}\Big[e^{-F(\theta,\bar{x},u)}\varphi(X^u_{\bar{x}}(\theta), e^{F(\theta,\bar{x},u)}\bar{y}) + \int_0^\theta f(X^u_{\bar{x}}(s), u)e^{-F(s,\bar{x},u)}ds\Big]$$

and the desired property can be obtained by standard passages, with the usual modifications required for proving the super-solution inequality. □

4.6 Comparison principle

After having shown that ϑ solves equation (4.5.9), we now consider the uniqueness question. As usual in viscosity solution theory, we establish uniqueness in form of a comparison principle between USC sub-solutions and LSC super-solutions. In proving such a comparison principle, some additional difficulties arise because of the degeneracy of f near $\mathcal{T}$. In order to overcome this difficulty we will show that for any super-solution (resp. sub-solution) a super-optimality (resp. sub-optimality) principle holds and then we will use this result for proving the comparison principle by a direct calculation. The proof of the optimality principles given here adapts the techniques in presented in [23, Theorem 2.32] to the particular case of the second order boundary value problem (4.5.9).

Let us start with a preliminary result. We can in fact prove that, thanks to assumption (H_h)-(ii), together with (H_b),(H_σ) and (H_f), for any control $u \in \mathcal{U}$ and $T \geq 0$, aside from the standard estimation for the process $X^u_x(\cdot)$

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|X^u_x(t) - x\big|^2\Big] \leq Ce^{CT}(1 + |x|^2)T \tag{4.6.1}$$

(see Proposition 2.1.1, Chapter 2) the following inequalities also holds for $Y_{x,y}(\cdot)$: if $(x, y) \in \overline{\mathcal{O}}$, then

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|Y^u_{x,y}(t) - y\big|^2\Big] \leq Ce^{CT}\Big(|1 - e^{M_fT}|^2 + e^{2M_fT}(1 + |x|^2)T\Big) \tag{4.6.2}$$

where C is a constant depending only on the Lipschitz constants of b and σ, and M_f denotes the bound of the function f. We prove the following result for a later use.

Lemma 4.6.1. *For any $\varepsilon > 0, T \geq 0$ and $(x,y) \in \overline{\mathcal{O}}$ one has*

$$\sup_{u\in\mathcal{U}} \mathbb{P}\bigg[\sup_{t\in[0,T]} |(X_x^u(t), Y_{x,y}^u(t)) - (x,y)| > \varepsilon\bigg] \leq \frac{C_{T,x}}{\varepsilon^2}$$

for $C_{T,x} := Ce^{CT}(1 + e^{2M_f T})(T + |1 - e^{M_f T}|^2)(1 + |x|^2)$.

Proof. The result is a consequence of Doob's inequality applied to the sub-martingale $M_t := \sup_{s\in[0,t]}(|X(s) - x| + |Y(s) - y|)$ and of inequalities (4.6.1) and (4.6.2). Indeed for any $u \in \mathcal{U}$ one has:

$$\begin{aligned}
&\mathbb{P}\bigg[\sup_{t\in[0,T]} |(X_x^u(t), Y_{x,y}^u(t)) - (x,y)| > \varepsilon\bigg] \\
&\leq \mathbb{P}\bigg[\sup_{t\in[0,T]} (|X_x^u(t) - x| + |Y_{x,y}^u(t) - y|) > \varepsilon\bigg] \\
&\leq \frac{1}{\varepsilon^2}\,\mathbb{E}\bigg[\bigg(\sup_{t\in[0,T]} |X_x^u(t) - x| + |Y_{x,y}^u(t)) - y|\bigg)^2\bigg] \\
&\leq \frac{2}{\varepsilon^2}\,\mathbb{E}\bigg[\sup_{t\in[0,T]} |X_x^u(t) - x|^2 + \sup_{t\in[0,T]} |Y_{x,y}^u(t)) - y|^2\bigg] \\
&\leq \frac{Ce^{CT}}{\varepsilon^2}\,\Big((1 + e^{2M_f T})T(1 + |x|^2) + |1 - e^{M_f T}|^2\Big)
\end{aligned}$$

where C is the constant appearing in (4.6.1) and (4.6.2). This shows the claim. □

Let us define the domain

$$\mathcal{O}_\delta := \Big\{(x,y) \in \overline{\mathcal{O}} : d_T^+(x) > \delta,\ y < -\delta\Big\}$$

and the associated exit time for the process $(X_x^u(t), Y_{x,y}^u(t))$

$$\tau_\delta^u := \inf\Big\{t \geq 0 : (X_x^u(t), Y_{x,y}^u(t)) \notin \mathcal{O}_\delta\Big\}.$$

Theorem 4.6.2. *Let $\underline{\vartheta} \in USC(\overline{\mathcal{O}})$ be a bounded viscosity sub-solution to equation (4.5.9) such that*

$$\underline{\vartheta}(x,y) = 1 \qquad \text{on } \partial_1\mathcal{O}.$$

Then $\underline{\vartheta}$ satisfies

$$\begin{aligned}
\underline{\vartheta}(x,y) \leq \inf_{u\in\mathcal{U}} \mathbb{E}\bigg[&e^{-F(\tau_\delta^u(t),x,u)}\underline{\vartheta}(X_x^u(\tau_\delta^u(t)), Y_{x,y}^u(\tau_\delta^u(t))) \\
&+ \int_0^{\tau_\delta^u(t)} f(X_x^u(s), u(s))e^{-F(s,x,u)}ds\bigg]
\end{aligned} \tag{4.6.3}$$

for any $(x,y) \in \mathcal{O}_\delta$, $t \geq 0$, where $\tau_\delta^u(t) := \min(t, \tau_\delta^u)$ and τ_δ^u denotes the exit time of the process $(X_x^u(\cdot), Y_{x,y}^u(\cdot))$ from the domain $\mathcal{O}_\delta$.

Proof. Let us start observing that since $\underline{\vartheta}$ is upper semi-continuous we can write for any $(x, y) \in \overline{\mathcal{O}}$

$$\underline{\vartheta}(x, y) = \inf_{k \geq 0} V_k(x, y) \tag{4.6.4}$$

where $\{V_k\}_{k \geq 0}$ is a decreasing sequence of bounded continuous functions. Let us consider for $k \geq 0$ the following evolutionary obstacle problem

$$\begin{cases} \max\Big(\partial_t V + H(x, y, V, D_x V, \partial_y V, D_x^2 V)\ , V - V_k\Big) = 0 & (0, t] \times \mathcal{O} \\ V(t, x, y) = 1 & (0, t] \times \partial_1 \mathcal{O} \\ -\partial_y V(t, x, y) = 0 & (0, t] \times \partial_2 \mathcal{O} \\ V(0, x, y) = V_k(x, y) & \overline{\mathcal{O}}. \end{cases} \tag{4.6.5}$$

It is immediate to verify that $\underline{\vartheta}$ is a bounded viscosity sub-solution of this problem for any $k \geq 0$ and $t \geq 0$. For $t \geq 0$, we now define the following function

$$L^k(t, x, y) := \begin{cases} \inf\limits_{u \in \mathcal{U}} \mathbb{E}\Big[e^{-F(\tau_\delta^u(t), x, u)} V_k(X_x^u(\tau_\delta^u(t)), Y_{x,y}^u(\tau_\delta^u(t))) & \overline{\mathcal{O}_\delta} \\ \qquad + \int_0^{\tau_\delta^u(t)} f(X_x^u(s), u(s)) e^{-F(s, x, u)} ds \Big] & \\ V_k(x, y) & \overline{\mathcal{O}} \setminus \overline{\mathcal{O}_\delta}. \end{cases}$$

We are going to prove that the lower semi-continuous envelop of L^k is a viscosity super-solution of the obstacle problem (4.6.5).
Let us start proving that L^k is continuous in $t = 0$. Of course, we only need to prove the result in $\overline{\mathcal{O}}_\delta$. Noting that $L^k(0, x, y) = V_k(x, y)$ for any $u \in \mathcal{U}$ and $(x, y) \in \overline{\mathcal{O}}_\delta$, one has for any $(\xi, \eta) \in \overline{\mathcal{O}}_\delta$

$$\begin{aligned} &\Bigg| \mathbb{E}\Big[e^{-F(\tau_\delta^u(t), \xi, u)} V_k(X_\xi^u(\tau_\delta^u(t)), Y_{\xi,\eta}^u(\tau_\delta^u(t))) \\ &\quad + \int_0^{\tau_\delta^u(t)} f(X_\xi^u(s), u(s)) e^{-F(s, \xi, u)} ds \Big] - L^k(0, x, y) \Bigg| \\ &\leq \mathbb{E}\Big[|e^{-F(\tau_\delta^u(t), \xi, u)} V_k(X_\xi^u(\tau_\delta^u(t)), Y_{\xi,\eta}^u(\tau_\delta^u(t))) - V_k(x, y)| \Big] + M_f t \\ &\leq \mathbb{E}\Big[e^{-F(\tau_\delta^u(t), \xi, u)} |V_k(X_\xi^u(\tau_\delta^u(t)), Y_{\xi,\eta}^u(\tau_\delta^u(t))) - V_k(x, y)| \Big] \\ &\quad + \mathbb{E}\Big[|V_k(x, y)| \Big(1 - e^{-F(\tau_\delta^u(t), \xi, u)}\Big) \Big] + M_f t \end{aligned}$$

$$\le \mathbb{E}\Big[|V_k(X^u_\xi(\tau^u_\delta(t)), Y^u_{\xi,\eta}(\tau^u_\delta(t))) - V_k(\xi,\eta)|\Big] + |V_k(\xi,\eta) - V_k(x,y)|$$
$$+ C\Big(1 - e^{-M_f t}\Big) + M_f t.$$

Thanks to the continuity of V_k, there exists δ_ε such that

$$|V_k(x,y) - V_k(\xi,\eta)| \le \frac{\varepsilon}{4}$$

for any $(\xi,\eta) \in B((x,y),\delta_\varepsilon)$. Therefore if we define the set

$$E := \Big\{\omega \in \Omega : |\big(X^u_\xi(\tau^u_\delta(t)), Y^u_{\xi,\eta}(\tau^u_\delta(t))\big) - (\xi,\eta)| > \delta_\varepsilon\Big\}$$

we obtain

$$\int_{E^C} |V_k(X^u_\xi(\tau^u_\delta(t)), Y^u_{\xi,\eta}(\tau^u_\delta(t))) - V_k(\xi,\eta)| d\mathbb{P} \le \frac{\varepsilon}{4}.$$

Moreover, thanks to the boundedness of V_k we get

$$\int_E |V_k(X^u_\xi(\tau^u_\delta(t)), Y^u_{\xi,\eta}(\tau^u_\delta(t))) - V_k(\xi,\eta)| d\mathbb{P} \le 2M\mathbb{P}[E].$$

Using the result in Lemma 4.6.1 we can state that there exists a constant C such that

$$\mathbb{P}[E] \le \frac{Ce^{Ct}}{\delta_\varepsilon^2}(1 + e^{2M_f t})(t + |1 - e^{M_f t}|^2)(1 + |\xi|^2).$$

Therefore, there exists $t_\varepsilon > 0$ such that for $t < t_\varepsilon$

$$2M\mathbb{P}[E] + C\Big(1 - e^{-M_f t}\Big) + M_f t \le \frac{\varepsilon}{2}.$$

In conclusion we have proved that for any $\varepsilon > 0$, if $t < t_\varepsilon$ and $|(x,y) - (\xi,\eta)| < \delta_\varepsilon$

$$|L^k(t,\xi,\eta) - L^k(0,x,y)| \le \varepsilon$$

which proves continuity of L^k in $t = 0$.

Denoting by L^k_* the lower semi-continuous envelope of L^k, it is possible to prove that the following DPP holds (see [27, Theorem 4.3]):

$$\inf_{u\in\mathcal{U}} \mathbb{E}\Big[\int_0^{\tau^u_\delta(\theta)} f(X^u_x(s), u(s))e^{-F(s,x,u)}ds + \mathbb{1}_{\{\theta\ge\tau^u_\delta\}} V_k(X^u_x(\tau^u_\delta), Y^u_{x,y}(\tau^u_\delta))e^{F(\tau^u_\delta,x,u)}$$
$$+ \mathbb{1}_{\{\theta<\tau^u_\delta\}} L^k_*(t-\theta, X^u_x(\theta), Y^u_{x,y}(\theta))e^{F(\theta,x,u)}\Big] \le L^k(t,x,y).$$

for any stopping time $0 \le \theta \le t$.

Thanks to this result, applying the standard dynamic programming arguments (see for instance the proof given in [27]) and recalling that $L^k(t,x,y) = V_k(x,y)$ in $\overline{\mathcal{O}}\backslash\overline{\mathcal{O}_\delta}$, one has that L^k_* is a viscosity super-solution of (4.6.5) in $\overline{\mathcal{O}}$. We point out that, because of the continuity of L^k_* in $t = 0$ and $y = 0$, the initial condition and the boundary condition on $\partial_1\mathcal{O}$ are satisfied in the strong sense.

We will prove in Section 4.6.1 that for equation (4.6.5) a comparison principle for semi-continuous viscosity sub- and super-solution holds. It can be obtained by the arguments in [93] adapted to the parabolic case (see also Section 3.10 in Chapter 3). The necessity of using such a result instead of a more classical comparison principle for fully nonlinear second order elliptic equations with oblique derivative boundary conditions, as that one presented for instance in [84] (see also the references therein), comes from the lack of regularity of the domain $\overline{\mathcal{O}}$. Since the key arguments of the proof in [93] easily extend to our context, in Section 4.6.1 we will only give a sketch of the proof. Applying Theorem 4.6.6, we obtain for any $(t,x,y) \in [0,+\infty) \times \mathcal{O}_\delta$

$$\underline{\vartheta}(x,y) \leq L^k_*(t,x,y),$$

which leads to

$$\begin{aligned}\underline{\vartheta}(x,y) \leq \mathbb{E}\Big[& e^{-F(\tau^u_\delta(t),x,u)} V_k(X^u_x(\tau^u_\delta(t)), Y^u_{x,y}(\tau^u_\delta(t))) \\ &+ \int_0^{\tau^u_\delta(t)} f(X^u_x(s),u(s)) e^{-F(s,x,u)} ds\Big] \qquad \forall u \in \mathcal{U}.\end{aligned}$$

It remains to pass to the limit for $k \to +\infty$. Recalling expression (4.6.4) for $\underline{\vartheta}$ we have

$$\underline{\vartheta}(x,y) = \limsup_{k\to+\infty} V_k(x,y)$$

and then for any $u \in \mathcal{U}$

$$\begin{aligned}&\underline{\vartheta}(x,y) \\ &\leq \limsup_{k\to+\infty} \mathbb{E}\Big[e^{-F(\tau^u_\delta(t),x,u)} V_k(X^u_x(\tau^u_\delta(t)), Y^u_{x,y}(\tau^u_\delta(t))) \\ &\qquad\qquad + \int_0^{\tau^u_\delta(t)} f(X^u_x(s),u(s)) e^{-F(s,x,u)} ds\Big] \\ &\leq \mathbb{E}\Big[\limsup_{k\to+\infty} e^{-F(\tau^u_\delta(t),x,u)} V_k(X^u_x(\tau^u_\delta(t)), Y^u_{x,y}(\tau^u_\delta(t))) \\ &\qquad\qquad + \int_0^{\tau^u_\delta(t)} f(X^u_x(s),u(s)) e^{-F(s,x,u)} ds\Big] \\ &= \mathbb{E}\Big[e^{-F(\tau^u_\delta(t),x,u)} \underline{\vartheta}(X^u_x(\tau^u_\delta(t)), Y^u_{x,y}(\tau^u_\delta(t))) + \int_0^{\tau^u_\delta(t)} f(X^u_x(s),u(s)) e^{-F(s,x,u)} ds\Big]\end{aligned}$$

where for the second inequality we used Fatou's lemma, thanks to the boundedness of the functions V_k. Hence, the desired result is obtained thanks to the arbitrariness of $u \in \mathcal{U}$. □

The same techniques can be applied in order to prove the super-optimality principle for LSC super-solutions. In this case, however, compactness assumption on the dynamics (considering weak solutions of the SDE) are necessary in order to guarantee the last passage to the limit (see [80]). The version of the super-optimality principle we state below avoids this kind of assumption by taking into account only continuous super-solutions.

Theorem 4.6.3. *Let $\overline{\vartheta} \in C(\overline{\mathcal{O}})$ be a bounded viscosity super-solution to equation (4.5.9). Then for any $(x, y) \in \mathcal{O}_\delta$ and $t \geq 0$*

$$\overline{\vartheta}(x,y) \geq \inf_{u\in\mathcal{U}} \mathbb{E}\Big[e^{-F(\tau^u_\delta(t),x,u)}\overline{\vartheta}(X^u_x(\tau^u_\delta(t)), Y^u_{x,y}(\tau^u_\delta(t))) + \int_0^{\tau^u_\delta(t)} f(X^u_x(s), u(s)) e^{-F(s,x,u)} ds \Big]. \tag{4.6.6}$$

Proof. Let us consider the following evolutionary obstacle problem:

$$\begin{cases} \min\Big(\partial_t V + H(x,y,V,D_xV,\partial_yV,D^2_xV)\,, V-\overline{\vartheta}\Big) = 0 & (0,t]\times\mathcal{O} \\ V(t,x,y) = 1 & (0,t]\times\partial_1\mathcal{O} \\ -\partial_y V(t,x,y) = 0 & (0,t]\times\partial_2\mathcal{O} \\ V(0,x,y) = \overline{\vartheta}(x,y) & \overline{\mathcal{O}}. \end{cases} \tag{4.6.7}$$

We can easily observe that $\overline{\vartheta}$ is a viscosity super-solution to (4.6.7). In what follows, we build a viscosity sub-solution for problem (4.6.7). Let $W : \overline{\mathcal{O}} \to \mathbb{R}$ be defined by

$$W(t,x,y) := \begin{cases} \inf_{u\in\mathcal{U}} \mathbb{E}\Big[e^{-F(\tau^u_\delta(t),x,u)}\overline{\vartheta}(X_x(\tau_\delta(t)), Y_{x,y}(\tau_\delta(t))) & \text{in } \overline{\mathcal{O}}_\delta \\ \qquad + \int_0^{\tau^u_\delta(t)} f(X^u_x(s), u(s)) e^{-F(s,x,u)} ds \Big] & \\ \overline{\vartheta}(x,y) & \text{in } \overline{\mathcal{O}} \setminus \overline{\mathcal{O}}_\delta \end{cases}$$

Let us consider its upper semi-continuous envelope W^*. By similar arguments as in Theorem 4.6.2 we can prove that W^* is a viscosity sub-solution to (4.6.7). Indeed, the continuity with respect to time in $t = 0$ can be prove as in Theorem 4.6.2. Moreover, the boundary condition on $\partial_1\mathcal{O}$ is satisfied in the strong sense thanks to the continuity of $\overline{\vartheta}$. Therefore, applying the comparison principle Theorem 4.6.6 between sub and super solutions to (4.6.7) we get

$$\overline{\vartheta}(x,y) \geq W^*(t,x,y).$$

This yields

$$\overline{\vartheta}(x,y) \geq \inf_{u\in\mathcal{U}} \mathbb{E}\Big[e^{-F(\tau^u_\delta(t),x,u)}\overline{\vartheta}(X^u_x(\tau^u_\delta(t)), Y^u_{x,y}(\tau^u_\delta(t))) + \int_0^{\tau^u_\delta(t)} f(X^u_x(s)) e^{-F(s,x,u)} ds \Big]$$

for any $t \geq 0, (x,y) \in \overline{\mathcal{O}}_\delta$. □

The super-optimality principle from Theorem 4.6.3 and the sub-optimality principle from Theorem 4.6.2 are finally used in the next theorem in order to establish the desired comparison result.

Theorem 4.6.4. *Let $\underline{\vartheta} \in USC(\overline{\mathcal{O}})$ and $\overline{\vartheta} \in C(\overline{\mathcal{O}})$ be a bounded viscosity sub- and super-solution to equation (4.5.9), respectively. Let us also assume that*

$$\underline{\vartheta}(x,y) \leq 1 + y \leq \overline{\vartheta}(x,y) \quad \text{on} \quad \{(x,y) \in \overline{\mathcal{O}} : x \in \mathcal{T}\} \tag{4.6.8}$$

and

$$\underline{\vartheta}(x,0) = \overline{\vartheta}(x,0) = 1 \qquad \forall x \in \overline{\mathcal{O}}. \tag{4.6.9}$$

Then $\underline{\vartheta}(x,y) \leq \overline{\vartheta}(x,y)$ for any $(x,y) \in \overline{\mathcal{O}}$.

Proof. Clearly, if $x \in \mathcal{T}$ there is nothing to prove. Thanks to inequalities (4.6.6) and (4.6.3), for any $(x, y) \in \mathcal{O}_\delta$ and $T \geq 0$ we have

$$
\begin{aligned}
&\underline{\vartheta}(x,y) - \overline{\vartheta}(x,y) \\
&\leq \sup_{u \in \mathcal{U}} \mathbb{E}\Big[e^{-F(\tau_\delta^u(T),x,u)}\Big(\underline{\vartheta}(X_x^u(\tau_\delta^u(T)), Y_{x,y}^u(\tau_\delta^u(T))) - \overline{\vartheta}(X_x^u(\tau_\delta^u(T)), Y_{x,y}^u(\tau_\delta^u(T)))\Big)\Big] \\
&= \sup_{u \in \mathcal{U}}\Big\{ \int_{\tau_\delta^u \leq T} e^{-F(\tau_\delta,x,u)}\Big(\underline{\vartheta}(X_x^u(\tau_\delta^u), Y_{x,y}^u(\tau_\delta^u)) - \overline{\vartheta}(X_x^u(\tau_\delta^u), Y_{x,y}^u(\tau_\delta^u))\Big) d\mathbb{P} \\
&\quad + \int_{\tau_\delta^u > T} e^{-F(T,x,u)}\Big(\underline{\vartheta}(X_x^u(T), Y_{x,y}^u(T)) - \overline{\vartheta}(X_x^u(T), Y_{x,y}^u(T))\Big) d\mathbb{P}\Big\}
\end{aligned}
$$

We will study these two integrals separately. Thanks to the (semi-)continuity of $\overline{\vartheta}$ and $\underline{\vartheta}$ and conditions (4.6.8) and (4.6.9), for any $\varepsilon > 0$ it is possible to find δ_ε small enough such that

$$
\underline{\vartheta}(x,y) \leq 1 + y + \frac{\varepsilon}{2}, \qquad \overline{\vartheta}(x,y) \geq 1 + y - \frac{\varepsilon}{2} \qquad \text{if} \quad d_{\mathcal{T}}^+(x) \leq \delta_\varepsilon
$$

and

$$
\underline{\vartheta}(x,y) \leq 1 + \frac{\varepsilon}{2}, \qquad \overline{\vartheta}(x,y) \geq 1 - \frac{\varepsilon}{2} \qquad \text{if} \quad y \geq -\delta_\varepsilon.
$$

Recalling that τ_δ^u is the exit time from the domain $\mathcal{O}_\delta$, we have that for any $u \in \mathcal{U}$ either $Y_{x,y}^u(\tau_\delta^u) = -\delta$ or $d(X_x^u(\tau_\delta^u), \mathcal{T}) = \delta$. For both these cases, choosing δ small enough, for the first integral we find

$$
\begin{aligned}
&\int_{\tau_\delta^u \leq T} e^{-F(\tau_\delta^u,x,u)}\Big(\underline{\vartheta}(X_x^u(\tau_\delta^u), Y_{x,y}^u(\tau_\delta^u)) - \overline{\vartheta}(X_x^u(\tau_\delta^u), Y_{x,y}^u(\tau_\delta^u))\Big) d\mathbb{P} \\
&\leq \varepsilon\, \mathbb{P}[\tau_\delta^u \leq T] \leq \varepsilon.
\end{aligned}
$$

For the second integral we can use the boundedness of $\overline{\vartheta}$ and $\underline{\vartheta}$. Denoting by M a bound for these functions, we obtain for any $u \in \mathcal{U}$

$$
\begin{aligned}
&\int_{\tau_\delta^u > T} e^{-F(T,x,u)}\Big(\underline{\vartheta}(X_x^u(T), Y_{x,y}^u(T)) - \overline{\vartheta}(X_x^u(T), Y_{x,y}^u(T))\Big) d\mathbb{P} \\
&\leq 2M \int_{\tau_\delta^u > T} e^{-F(T,x,u)} d\mathbb{P}.
\end{aligned}
$$

If we define

$$
f^* := \inf\Big\{ f(x,u) \;\Big|\; x \in \mathbb{R} : d_{\mathcal{T}}^+(x) > \delta, u \in U \Big\} > 0
$$

we finally obtain for T large enough

$$
\underline{\vartheta}(x,y) - \overline{\vartheta}(x,y) \leq \; \varepsilon + e^{-f^* T} = 2\varepsilon
$$

for any $(x, y) \in \mathcal{O}_\delta$ and the result is obtained thanks to the arbitrariness of ε.

Finally. we obtain the desired comparison principle in the whole domain $\overline{\mathcal{O}}$ by sending $\delta \to 0$, thanks to the upper semi-continuity of the function $\underline{\vartheta} - \overline{\vartheta}$. □

An immediate consequence of this theorem and Theorem 4.5.4 is the following existence and uniqueness result.

Corollary 4.6.5. *Let assumptions (H_b),(H_σ),$(H'_{\mathcal{T}})$,$(H'_{\mathcal{K}})$,(H_f) and (H_h) be satisfied. Then ϑ from (4.5.1) is the unique bounded and continuous viscosity solution to equation (4.5.9) such that $\vartheta(x,y) = 1 + y$ if $x \in \mathcal{T}$ and $\vartheta(x,0) = 1$ for any $x \in \mathbb{R}^d$.*

4.6.1 Comparison principle for obstacle problems with Dirichlet-Neumann boundary conditions

In this section we will give a proof of a comparison principle for the obstacle problems (4.6.5) and (4.6.7). Before stating the result and starting its proof we introduce a more compact notation. Let us start defining

$$\tilde{b}(x,y,u) := \begin{pmatrix} b(x,u) \\ yf(x,u) \end{pmatrix} \in \mathbb{R}^{d+1} \quad \text{and} \quad \tilde{\sigma}(x,y,u) := \begin{pmatrix} \sigma(x,u) \\ 0 \dots 0 \end{pmatrix} \in \mathbb{R}^{(d+1)\times p}.$$

In what follow we will directly denote with x the variable in the augmented state space $\mathbb{R}^n$ for $n := d+1$, that is $x \equiv (x,y) \in \mathbb{R}^n$. Using this notation we can write the Hamiltonian H in (4.5.8) in the compact form

$$H(x,r,q,Q) := \sup_{u\in U}\Big\{ -q\cdot\tilde{b}(x,u) - \frac{1}{2}Tr[\tilde{\sigma}\tilde{\sigma}^T(x,u)Q] + f(x,u)(r-1)\Big\}.$$

The boundary value problem we deal with is the following

$$\begin{cases} \min\Big(\partial_t V + H(x,V,DV,D^2V)\,, V-\psi\Big) = 0 & (0,T)\times\mathcal{O} \\ V(t,x) = 1 & (0,T)\times\partial_1\mathcal{O} \\ -\partial_{x_n}V(t,x) = 0 & (0,T)\times\partial_2\mathcal{O} \\ V(0,x) = \psi(x) & \overline{\mathcal{O}} \end{cases} \tag{4.6.10}$$

(where ∂_{x_n} denotes the partial derivative with respect to the n-th space variable and $\psi(x) = 1$ on $\partial_1\mathcal{O}$). We recall that the boundary conditions in $t=0$ and $\partial_1\mathcal{O}$ are considered in the strong sense, that is for any viscosity sub-solution $\underline{V}$ (resp. super-solution $\overline{V}$) one has

$$\underline{V}(0,x) \le \psi(x) \qquad (\text{resp. } \overline{V}(0,x) \ge \psi(x)) \qquad \text{on } \overline{\mathcal{O}}$$

and

$$\underline{V}(t,x) \le 1 \qquad (\text{resp. } \overline{V}(t,x) \ge 1) \qquad \text{on } (0,T)\times\partial_1\mathcal{O}.$$

We also recall that on the boundary $\partial_2\mathcal{O}$ the following weak conditions

$$\min\Big(\min\big(\partial_t\underline{V} + H(x,\underline{V},D\underline{V},D^2\underline{V}),\underline{V}-\psi\big)\,,\,-\partial_{x_n}\underline{V}\Big) \le 0$$

$$\max\Big(\min\big(\partial_t\overline{V} + H(x,\overline{V},D\overline{V},D^2\overline{V}),\overline{V}-\psi\big)\,,\,-\partial_{x_n}\overline{V}\Big) \ge 0$$

are considered, in the viscosity sense, respectively, for sub- and super-solutions.
In the sequel we will denote by $|\cdot|_{n-1}$ the norm restricted to the first $n-1$ components of the vector, that is:

$$|x|_{n-1} := |(x_1,\dots,x_{n-1})|, \qquad \forall x\in\mathbb{R}^n.$$

Theorem 4.6.6. *Assume* (H_b),(H_σ),(H_f),(H_h),$(H'_\mathcal{K})$ *and* $\psi \in C(\overline{\mathcal{O}})$. *Let* $\underline{V} \in USC([0,T]\times\overline{\mathcal{O}})$ *and* $\overline{V} \in LSC([0,T]\times\overline{\mathcal{O}})$ *be respectively a bounded viscosity sub- and super- solution to* (4.6.10)*. Then for any* $x\in\overline{\mathcal{O}}$ *and* $t\in[0,T)$

$$\underline{V}(t,x) \le \overline{V}(t,x).$$

Since the main arguments of the proof can be found in [93, Theorem 2.1] (some of them are also reported in Section 3.10, Chapter 3), below we only report the main lines.

Sketch of the proof of Theorem 4.6.6. Recalling that the boundary $\partial_2\mathcal{O}$ is defined by the function $-e^{-h(x_1,\dots,x_{n-1})}$, thanks to the Lipschitz assumption (H_h)-(ii), we can easily observe that taking $\mu := 1/\sqrt{1+L_h^2}$, where L_h is the Lipschitz constant appearing in (H_h)-(ii), for any $z \in \partial_2\mathcal{O}$ one has

$$\bigcup_{0\le\xi\le\mu} B(z-\xi,\xi\mu) \subset \mathcal{O}^C. \tag{4.6.11}$$

This corresponds to condition (2.9) in [93] and by the same arguments as in Lemma 3.10.3, Chapter 3 the existence of a function $\zeta \in C^2(\overline{\mathcal{O}})$ follows such that $\zeta \ge 0$ on $\overline{\mathcal{O}}$, $-\partial_{x_n}\zeta \ge 1$ on $\partial_2\mathcal{O}$ and $|D\zeta|, \|D^2\zeta\| \le K_\zeta$ for some constant $K_\zeta \ge 0$.

Let us define for $\delta,\rho,\beta > 0$

$$\underline{V}_{\delta,\rho,\beta}(t,x) := \underline{V}(t,x) - \delta e^{-\rho T}\zeta(x) - \frac{\beta}{T-t}$$

and

$$\overline{V}_{\delta,\rho,\beta}(t,x) := \overline{V}(t,x) + \delta e^{-\rho T}\zeta(x) + \frac{\beta}{T-t}.$$

One has

$$\underline{V}_{\delta,\rho,\beta} \stackrel{t\to T}{\longrightarrow} -\infty \quad \text{and} \quad \overline{V}_{\delta,\rho,\beta} \stackrel{t\to T}{\longrightarrow} +\infty.$$

It is possible to verify that $\underline{V}_{\delta,\rho,\beta}$ (resp. $\overline{V}_{\delta,\rho,\beta}$) is a sub-solution (resp. super-solution) of an obstacle problem with the following modified boundary conditions on $\partial_2\mathcal{O}$:

$$-\partial_{x_n}V + \delta e^{-\rho T} \le 0 \qquad \left(\text{resp. } -\partial_{x_n}V - \delta e^{-\rho T} \ge 0\right). \tag{4.6.12}$$

Moreover, thanks to the positivity of ζ, one has

$$\underline{V}_{\delta,\rho,\beta} \le \underline{V} \quad \text{and} \quad \overline{V}_{\delta,\rho,\beta} \ge \overline{V},$$

so the boundary conditions for $t=0$ and $y=0$ in (4.6.7) are trivially satisfied. By using the non negativity of f and the linear growth of $\tilde{b}$ and $\tilde{\sigma}$, in $\overline{\mathcal{O}}$ one has

$$\begin{aligned}
&H(x,\underline{V}_{\delta,\rho,\beta},D\underline{V}_{\delta,\rho,\beta},D^2\underline{V}_{\delta,\rho,\beta}) - H(x,\underline{V},D\underline{V},D^2\underline{V})\\
&\le H(x,\underline{V},D\underline{V}_{\delta,\rho,\beta},D^2\underline{V}_{\delta,\rho,\beta}) - H(x,\underline{V},D\underline{V},D^2\underline{V})\\
&\le \sup_{u\in U}\left|\tilde{b}(x,u)\cdot\delta e^{-\rho T}D\zeta + \frac{1}{2}Tr[\tilde{\sigma}\tilde{\sigma}^T(x,u)(\delta e^{-\rho T}D^2\zeta)]\right|\\
&\le C_1\delta e^{-\rho T}(1+|x|_{n-1}^2),
\end{aligned}$$

where C_1 only depends on K_ζ and the Lipschitz constants of b and σ. Then if

$$\min\left(\partial_t\underline{V} + H(x,\underline{V},D\underline{V},D^2\underline{V})\ ,\ \underline{V} - \psi(x)\right) \le 0$$

for some $x \in \mathcal{O}\cup\partial_2\mathcal{O}$ one obtains

$$\begin{aligned}
&\min\Big(\partial_t\underline{V}_{\delta,\rho,\beta} + H(x,\underline{V}_{\delta,\rho,\beta},D\underline{V}_{\delta,\rho,\beta},D^2\underline{V}_{\delta,\rho,\beta}) + \frac{\beta}{T^2} - C_1\delta e^{-\rho T}(1+|x|_{n-1}^2)\ ,\\
&\qquad \underline{V}_{\delta,\rho,\beta} + \frac{\beta}{T} - \psi(x)\Big)\\
&\le \min\left(\partial_t\underline{V} + H(x,\underline{V},D\underline{V},D^2\underline{V})\ ,\ \underline{V} - \psi(x)\right) \le 0.
\end{aligned}$$

The analogous result can be proved for the super-solution $\overline{V}_{\delta,\rho,\beta}$.

Our goal is now to prove the inequality

$$\underline{V}_{\delta,\rho,\beta}(t,x) \leq \overline{V}_{\delta,\rho,\beta}(t,x) + 2C_1\delta e^{-\rho(T-t)}(1+|x|^2_{n-1}) \tag{4.6.13}$$

for all $\delta, \rho, \beta > 0$. By virtue of the definition of $\underline{V}_{\delta,\rho,\beta}$ and $\overline{V}_{\delta,\rho,\beta}$, this implies the claim of the theorem by letting $\delta, \beta \to 0$.

In order to prove (4.6.13), we consider the modified obstacle problems given by

$$\begin{cases} \min\Big(\partial_t V + H(x,V,DV,D^2V) + \frac{\beta}{T^2} - C_1\delta e^{-\rho T}(1+|x|^2_{n-1})\ , & \\ \qquad V + \frac{\beta}{T} - \psi\Big) \leq 0 & (0,T)\times\mathcal{O} \\ V(t,x) \leq 1 & (0,T)\times\partial_1\mathcal{O} \\ -\partial_{x_n}V(t,x) + \delta e^{-\rho T} \leq 0 & (0,T)\times\partial_2\mathcal{O} \\ V(0,x) \leq \psi(x) & \overline{\mathcal{O}} \end{cases}$$

and

$$\begin{cases} \min\Big(\partial_t V + H(x,V,DV,D^2V) - \frac{\beta}{T^2} + C_1\delta e^{-\rho T}(1+|x|^2_{n-1})\ , & \\ \qquad V - \frac{\beta}{T} - \psi\Big) \geq 0 & (0,T)\times\mathcal{O} \\ V(t,x) \geq 1 & (0,T)\times\partial_1\mathcal{O} \\ -\partial_{x_n}V(t,x) - \delta e^{-\rho T} \geq 0 & (0,T)\times\partial_2\mathcal{O} \\ V(0,x) \geq \psi(x) & \overline{\mathcal{O}} \end{cases}$$

and consider the function

$$\Phi(t,x) := \underline{V}_{\delta,\rho,\beta}(t,x) - \overline{V}_{\delta,\rho,\beta}(t,x) - 2C_1\delta e^{-\rho(T-t)}(1+|x|^2_{n-1}).$$

Thanks to the boundedness and the semi-continuity of $\underline{V}_{\delta,\rho,\beta}$ and $\overline{V}_{\delta,\rho,\beta}$, Φ admits a maximum point $(\hat{t}_{\delta,\rho,\beta}, \hat{x}_{\delta,\rho,\beta}) = (\hat{t},\hat{x})$. If either $\hat{t} = 0$ or $\hat{x} \in \partial_1\mathcal{O}$, then (4.6.13) follows from the boundary conditions. Similarly, (4.6.13) follows immediately in case $\Phi(\hat{t},\hat{x}) \leq 0$. If $\hat{x} \in \mathcal{O}$, inequality (4.6.13) can be proved by using classical comparison results for obstacle problems, see [170, Theorem 7.8] (see also the discussion of Case 1 and 2 below).

It remains to consider the case $\hat{x} \in \partial_2\mathcal{O}$, for which we will show that it cannot occur if $\hat{t} > 0$, $\hat{x} \notin \partial_1\mathcal{O}$ and $\Phi(\hat{t},\hat{x}) > 0$ and if $\rho > 0$ is sufficiently large (observe that it is enough to establish (4.6.13) for all sufficiently large ρ because this will imply (4.6.13) for all $\rho > 0$). Thanks to the property (4.6.11) of our domain, the existence of a family of C^2 test functions $\{w_\varepsilon\}_{\varepsilon>0}$ as in [93, Theorem 4.1] can be proved (see also Lemma 3.10.2, Chapter 3). Among the other properties, $\{w_\varepsilon\}_{\varepsilon>0}$ satisfies:

$$w_\varepsilon(x,x) \leq \varepsilon \tag{4.6.14}$$

$$w_\varepsilon(x,y) \geq C\frac{|x-y|^2}{\varepsilon} \tag{4.6.15}$$

$$-\partial_{x_n}w_\varepsilon(x,y) \geq -C\frac{|x-y|^2}{\varepsilon} \quad \text{if } x \in \partial_2\mathcal{O}\cap B(\hat{x},\eta), y \in B(\hat{x},\eta) \tag{4.6.16}$$

$$-\partial_{y_n}w_\varepsilon(x,y) \geq 0 \quad \text{if } y \in \partial_2\mathcal{O}\cap B(\hat{x},\eta), x \in B(\hat{x},\eta) \tag{4.6.17}$$

for $\varepsilon > 0$ and some $\eta > 0$ small enough.

Applying the doubling variables procedure we define

$$\begin{aligned}\Phi_\varepsilon(t,x,y) :=&\underline{V}_{\delta,\rho,\beta}(t,x) - \overline{V}_{\delta,\rho,\beta}(t,y) - C_1\delta e^{-\rho(T-t)}(1+|x|^2_{n-1})\\ &- C_1\delta e^{-\rho(T-t)}(1+|y|^2_{n-1}) - w_\varepsilon(x,y) - |x-\hat{x}|^4 - |t-\hat{t}|^2\end{aligned}$$

and we denote by $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$ its maximum point. By the usual techniques, thanks to the properties (4.6.14) and (4.6.15), it is possible to prove that for ε going to 0

$$x_\varepsilon, y_\varepsilon \to \hat{x}, \qquad t_\varepsilon \to \hat{t} \qquad \text{and} \qquad \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \to 0.$$

It follows that for ε small enough we can assume that $x_\varepsilon, y_\varepsilon \notin \partial_1\mathcal{O}$ and $t_\varepsilon > 0$. Taking ε small enough we can also say that $x_\varepsilon, y_\varepsilon \in B(\hat{x},\eta)$ and then we can make use of properties (4.6.16) and (4.6.17). In particular if $x_\varepsilon \in \partial_2\mathcal{O}$, taking ε small enough, we have

$$\begin{aligned}&-\partial_{x_n}\Big(w_\varepsilon(x_\varepsilon,y_\varepsilon) + \delta e^{-\rho(T-t_\varepsilon)}(1+|x_\varepsilon|^2_{n-1}) + |x_\varepsilon-\hat{x}|^4\Big)\\ &\geq -C\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon} - 4|x_\varepsilon-\hat{x}|^2|x_{\varepsilon_n}-\hat{x}_n| > -\delta e^{-\rho T}.\end{aligned}$$

On the other hand if $y_\varepsilon \in \partial_2\mathcal{O}$

$$-\partial_{y_n}\Big(-w_\varepsilon(x_\varepsilon,y_\varepsilon) - C_1\delta e^{-\rho(T-t_\varepsilon)}(1+|y_\varepsilon|^2_{n-1})\Big) \leq 0 < \delta e^{-\rho T}.$$

This means that for sufficiently small values of ε, we can neglect the derivative boundary conditions in $x_\varepsilon, y_\varepsilon$ and only consider

$$\begin{aligned}&\min\Big(\partial_t\underline{V}_{\delta,\rho,\beta} + H(x_\varepsilon,\underline{V}_{\delta,\rho,\beta},D\underline{V}_{\delta,\rho,\beta},D^2\underline{V}_{\delta,\rho,\beta}) + \frac{\beta}{T^2} - C_1\delta e^{-\rho T}(1+|x_\varepsilon|^2_{n-1})\,,\\ &\qquad \underline{V}_{\delta,\rho,\beta} + \frac{\beta}{T} - \psi\Big) \leq 0\\ &\min\Big(\partial_t\overline{V}_{\delta,\rho,\beta} + H(y_\varepsilon,\overline{V}_{\delta,\rho,\beta},D\overline{V}_{\delta,\rho,\beta},D^2\overline{V}_{\delta,\rho,\beta}) - \frac{\beta}{T^2} + C_1\delta e^{-\rho T}(1+|y_\varepsilon|^2_{n-1})\,,\\ &\qquad \overline{V}_{\delta,\rho,\beta} - \frac{\beta}{T} - \psi\Big) \geq 0\end{aligned}$$

in the viscosity sense.

- Case 1: let us assume that

$$\underline{V}_{\delta,\rho,\beta}(t_\varepsilon,x_\varepsilon) + \frac{\beta}{T} - \psi(x_\varepsilon) \leq 0.$$

In this case we would get (since $\overline{V}_{\delta,\rho,\beta}(t_\varepsilon,y_\varepsilon) - \frac{\beta}{T} - \psi(y_\varepsilon) \geq 0$ always holds)

$$\underline{V}_{\delta,\rho,\beta}(t_\varepsilon,x_\varepsilon) - \overline{V}_{\delta,\rho,\beta}(t_\varepsilon,y_\varepsilon) + \frac{2\beta}{T} + \psi(y_\varepsilon) - \psi(x_\varepsilon) \leq 0.$$

For sufficiently small $\varepsilon > 0$, from $\Phi(\hat{t},\hat{x}) > 0$ we know that $\Phi_\varepsilon(t_\varepsilon,x_\varepsilon,y_\varepsilon) > 0$ and this implies $\underline{V}_{\delta,\rho,\beta}(t_\varepsilon,x_\varepsilon) - \overline{V}_{\delta,\rho,\beta}(t_\varepsilon,y_\varepsilon) > 0$, leading to a contradiction for ε going to 0.

- Case 2: let us assume that

$$\partial_t\underline{V}_{\delta,\rho,\beta} + H(x,\underline{V}_{\delta,\rho,\beta},D\underline{V}_{\delta,\rho,\beta},D^2\underline{V}_{\delta,\rho,\beta}) + \frac{\beta}{T^2} - C_1\delta e^{-\rho T}(1+|x_\varepsilon|^2_{n-1}) \leq 0.$$

It follows that

$$\partial_t \underline{V}_{\delta,\rho,\beta}(t_\varepsilon, x_\varepsilon) - \partial_t \overline{V}_{\delta,\rho,\beta}(t_\varepsilon, y_\varepsilon) - C_1 \delta e^{-\rho T}(1 + |x_\varepsilon|^2_{n-1} + |y_\varepsilon|^2_{n-1})$$
$$+ H(x_\varepsilon, \underline{V}_{\delta,\rho,\beta}, D\underline{V}_{\delta,\rho,\beta}, D^2\underline{V}_{\delta,\rho,\beta}) - H(y_\varepsilon, \overline{V}_{\delta,\rho,\beta}, D\overline{V}_{\delta,\rho,\beta}, D^2\overline{V}_{\delta,\rho,\beta}) \le -\frac{2\beta}{T^2}.$$

Using the properties of the Hamiltonian H and of the test function w_ε, we can find a constant C_2 such that at the limit for $\varepsilon \to 0$ one has

$$-\frac{2\beta}{T^2} \ge 2\rho\delta e^{-\rho(T-\hat{t})}(1 + |\hat{x}|^2_{n-1}) - C_2\delta e^{-\rho(T-\hat{t})}(1 + 2|\hat{x}|^2_{n-1}) - C_1\delta e^{-\rho T}(1 + 2|\hat{x}|^2_{n-1})$$
$$\ge \delta e^{-\rho(T-\hat{t})}(1 + |\hat{x}|^2_{n-1})(\rho - C_2 - C_1)$$

and a contradiction is obtained as soon as $\rho \ge (C_1 + C_2 + 1)$.

□

Chapter 5

New approach for state constrained stochastic optimal control problems

Publications of this chapter

O. Bokanowski, A. Picarelli and H. Zidani, *State constrained stochastic optimal control problems via reachability approach*, preprint.

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5.1 Introduction

This chapter deals with stochastic optimal control problems in presence of state constraints. Let us denote by $X^u_{t,x}(\cdot)$, as usual, the strong solution associated with a certain control $u \in \mathcal{U}$ of a controlled SDE. Let $T > 0$ be fixed. We aim to characterize and compute the value function v defined by the following optimal control problem

$$v(t,x) := \inf_{u\in\mathcal{U}} \Big\{ \mathbb{E}\Big[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds\Big] : X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.} \Big\}. \tag{5.1.1}$$

In the unconstrained case, $\mathcal{K} = \mathbb{R}^d$, by using the dynamic programming approach the function v can be characterized as the unique viscosity solution of a second order HJB equation (classical references are [132, 133, 99], see also Chapter 2 and the references therein). However it is evident that many practical applications are concerned with the case $\mathcal{K} \subsetneq \mathbb{R}^d$ where, for instance, $\mathcal{K}$ takes into account the presence of an obstacle, economical/physical constraints etc.
In presence of state constraints the characterization of v as a viscosity solution of an HJB equation becomes more complicated and it is essentially due to its loss of regularity on the boundary. A rich literature has been developed for dealing with state constrained optimal control problems and the associated HJB equation: we can refer to [160, 161,

101, 112, 77, 136] for the deterministic case and to [117, 131, 114, 60] for the stochastic case. In all this literature some further conditions are established in order to supply to the loss of information on the boundary.

The aim of this chapter is to provide an alternative way for characterizing and compute, in a very general setting, the value function associated with a state constrained optimal control problem, trying to avoid the issues mentioned above. The approach we propose essentially relies on two ideas: the first one is to re-write problem (5.1.1) as a state constrained stochastic target problem and the second one is to solve this problem by a level set approach by using an exact penalization technique for managing the state constraints.

In the deterministic case ($\sigma \equiv 0$) the same approach has been applied with success in [2]. The results proposed in [2] are based on the following characterization of the value function v

$$\begin{aligned} v(t,x) &= \inf\Big\{ z \in \mathbb{R} : (x,z) \in \text{Epigraph}(v(t,\cdot)) \Big\} \\ &= \inf\Big\{ z \in \mathbb{R} : \exists u \in \mathcal{U} \text{ s.t. } z \geq \psi(X^u_{t,x}(T)),\ X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \Big\}. \end{aligned}$$

(see also [78, 79, 16, 12]). Here $\ell \equiv 0$ just to make it as clear as possible. The equality above shows how any optimal control problem can be seen as a state constrained reachability problem where the target set is represented by the epigraph of the cost function ψ.

In the stochastic setting the same techniques do not apply directly. In order to provide the link between our optimal control problem and the reachability one, we will use the result by Bouchard and Dang contained in [58]. Adapting the arguments of this paper to the constrained case, the following characterization of the function v is then obtained

$$\begin{aligned} v(t,x) = \inf\Big\{ & z \in \mathbb{R} : \exists (u,\alpha) \in \mathcal{U} \times \mathcal{A} \text{ s.t.} \\ & \Big(Z^{\alpha}_{t,z}(T) \geq \psi(X^u_{t,x}(T)),\ X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \Big) \text{ a.s. } \Big\} \end{aligned} \tag{5.1.2}$$

where $\mathcal{A}$ is the set of square-integrable predictable processes with values in $\mathbb{R}^p$ and, if $\ell \equiv 0$, $Z^{\alpha}_{t,z}$ belongs to a suitable set of martingales whose existence is guaranteed by the Ito's representation theorem. The right side of (5.1.2) is a stochastic target problem as defined in [163] with the addition of state constraints.

Analogously to what we did in Chapter 3, we propose here to solve the target problem in (5.1.2) by a level set approach, that has the advantage to lead to a computation of the function v, managing the eventual state constraints by an exact penalization technique without any further controllability assumption.

The chapter is organized as follows: we introduce the problem and the main assumptions in Section 5.2. In Section 5.3 we provide the link between the optimal control problem (5.1.1) and a suitable stochastic target problem. Section 5.4 is devoted to the solution of the reachability problem by the level set method. This will lead to a characterization of the reachable set by a generalized HJB equation in Section 5.5. A comparison principle for the associated boundary value problem is proved in Section 5.6. In section 5.7 we consider the uncontrolled case deriving a PDE characterization for the associated cost. We discuss the application of our method to an example coming from the electricity market in Section 5.8. In the appendix, Section 5.9, we prove an existence result necessary for the development of our arguments.

5.2 Setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathbb{F}_t\}_{t\geq 0}$, $\mathbb{P}$-augmentation of the filtration generated by a p-dimensional Brownian motion $\mathcal{B}(\cdot)$ $(p \geq 1)$.
Given $T > 0$ and $0 \leq t \leq T$, the following system of controlled SDE's in $\mathbb{R}^d$ is considered

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))d\mathcal{B}(s) & s \in [t, T] \\ X(t) = x, \end{cases} \tag{5.2.1}$$

where $u \in \mathcal{U}$ set of progressively measurable processes with values in $U \subset \mathbb{R}^m$ $(m \geq 1)$. Even if not specified at each time, along the whole chapter we will work under the assumption (H_U) of compactness of the set U. For the coefficients b and σ of the equation we will assume (H_b) and (H_σ) be satisfied.
Let us consider two functions $\psi : \mathbb{R}^d \to \mathbb{R}$ and $\ell : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}$, namely the terminal and running cost, such that:

$$(H_{\psi,\ell}) \qquad \begin{cases} (i) \quad \psi, \ell \text{ are continuous functions;} \\ (ii) \quad \psi, \ell \geq 0 \\ (iii) \quad \exists L_\psi, L_\ell \geq 0 \text{ such that } \forall x, y \in \mathbb{R}^d, t \in [0, T], u \in U : \\ \qquad\qquad |\psi(x) - \psi(y)| \leq L_\psi |x - y| \\ \qquad\qquad |\ell(t, x, u) - \ell(t, y, u)| \leq L_\ell |x - y|; \end{cases}$$

Remark 5.2.1. The assumption of non negativity of ψ and ℓ is not restrictive. It is in fact sufficient that $\psi, \ell \geq -M$ for some $M \geq 0$ and, by just adding a positive constant, we can recover assumption $(H_{\psi,\ell})$.

Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a given set of state constraints, such that

$$(H_{\mathcal{K}}) \qquad \mathcal{K} \subseteq \mathbb{R}^d \text{ is a nonempty and closed set.}$$

In this chapter we deal with optimal control problems for a cost in the following form:

$$J(t, x, u) := \mathbb{E}\left[\psi\left(X^u_{t,x}(T)\right) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds\right] \tag{5.2.2}$$

asking that the state $X^u_{t,x}$ satisfies almost surely (a.s.) some state constraints

$$X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t, T].$$

In other words we aim to solve the following optimal control problem

$$v(t, x) := \inf_{u \in \mathcal{U}} \left\{ J(t, x, u) : X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t, T] \text{ a.s. } \right\} \tag{5.2.3}$$

(observe that the non negativity of ψ and ℓ implies $J(t, x, u) \geq 0$, for all $(t, x, u) \in [0, T] \times \mathbb{R}^d \times \mathcal{U}$).

It is well known that in the unconstrained case $(\mathcal{K} = \mathbb{R}^d)$, it is possible to characterize v, via a dynamic programming approach, as the unique continuous viscosity solution of a second order HJB equation. We can refer for instance to [132, 133, 99] and [63] (see also the references given in Chapter 2).
When state constraints are taken into account, that is $\mathcal{K} \subset \mathbb{R}^d$, this characterization

becomes more complicated and a rich literature dealing with the partial differential equation associated to (5.2.3) is now available. The main problems that arise dealing with state constrained optimal control problems are due to the loss of continuity of the value function v on the boundary $\partial\mathcal{K}$. In absence of further hypotheses, v can only be characterized as a discontinuous viscosity solution of a state constrained HJB equation

$$\begin{cases} F(t,x,\partial_t v, Dv, D^2 v) = 0 & t \in [0,T), x \in int(\mathcal{K}) \\ F(t,x,\partial_t v, Dv, D^2 v) \geq 0 & t \in [0,T), x \in \partial\mathcal{K} \\ v(T,x) = \psi(x) & x \in \mathcal{K}. \end{cases}$$

with

$$F(t,x,r,q,Q) := -r + \sup_{u\in U}\Big\{ -q\cdot b(t,x,u) - \frac{1}{2}Tr[\sigma\sigma^T(t,x,u)Q] - \ell(t,x,u)\Big\}$$

and uniqueness cannot be proved. In order to overcome this difficulty several authors introduced in the years some assumptions in order to guarantee the continuity of the value function v. The first one that appeared in literature, in the context of deterministic control systems, is the so-called inward pointing condition, originally stated by Soner in [160] and [161]. Other references for first order equations are [112, 167, 166, 101, 100].

In the stochastic setting, $\sigma \not\equiv 0$, state constrained problems where introduced in [131]. In this work the diffusion is the identity matrix, so the presence of state constraints requires to consider unbounded controls, so that the trajectories with the action of the drift can still be maintained inside the desired domain. The result in that case is an HJB equation with singular boundary conditions. In our framework, the requirement that some state constraint is satisfied almost surely is strictly connected with some degeneracy property of the diffusion term. Important contributions in this case come from [117, 27, 35, 114] and, more recently, from [62]. In all these references it turns out the necessity of imposing on the boundary $\partial\mathcal{K}$ some conditions on the coefficients b and σ (together with some other more or less restrictive degeneracy assumptions on σ). We also point out that, in order to avoid that the value function v takes value $+\infty$, it is necessary at least the viability of the set $\mathcal{K}$. An example of the type of condition that guarantee such a property is given in Chapter 2 Section 2.5.

Our purpose in this chapter is to provide an alternative way for dealing with state constrained optimal control problems as (5.2.3), leading to the computation of v even in the case when this controllability assumptions are not necessarily satisfied. This approach, as we are going to detail in the next section, is strongly based on the existing equivalence between optimal control problems and reachability.

5.3 The associated reachability problem

The first step in our approach is to provide an equivalence between the optimal control problem (5.2.3) and a suitable target problem.

If $\sigma \equiv 0$, i.e. a deterministic setting is considered, it is now well-known that a sort of duality subsists between any optimal control problem and a reachability one (see [12, 16, 79] and [2]). It is fact not difficult to prove that, under our assumptions,

$$v(t,x) = \inf\Big\{ z \geq 0 : \exists u \in \mathcal{U} \text{ s.t.} \\ (X^u_{t,x}(T), Z^u_{t,x,z}(T)) \in \mathcal{T} \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \Big\} \tag{5.3.1}$$

where
$$\mathcal{T} := \{(x,z) \in \mathbb{R}^{d+1} : z \geq \psi(x)\} \equiv \text{Epigraph}(\psi)$$
and
$$Z^u_{t,z}(\cdot) := z - \int_t^{\cdot} \ell(s, X^u_{t,x}(s), u(s))ds.$$
We point out that this kind of formulation for optimal control problems had been used in [16, 12, 78, 79] for computing the value function v, or better its epigraph, by using viability tools (see [21, 15]).
Given a generic volatility $\sigma \not\equiv 0$, a relation analogous to (5.3.1) cannot be directly obtained and some additional steps are necessary. In particular in order to obtain the desired result we need to apply the arguments presented, for the unconstrained case, in [58].

We start by proving the following simple lemma:

Lemma 5.3.1. *Let assumptions* (H_b),(H_σ) *and* ($H_{\psi,\ell}$) *be satisfied. Then*

$$v(t,x) = \inf\Big\{ z \geq 0 : \exists u \in \mathcal{U} \text{ such that } J(t,x,u) \leq z \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.} \Big\} \tag{5.3.2}$$

Proof. If for any $u \in \mathcal{U}$ the state constraints is not almost surely satisfied there is nothing to prove. So let us assume that for any (t,x) there exists at least one control $u \in \mathcal{U}$ such that $X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T]$ almost surely.
By the definition of the value function v one has
$$v(t,x) \leq J(t,x,u), \qquad \forall u \in \mathcal{U} : X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.}.$$
It follows that for any $z \geq 0$ for which there exists $\hat{u} \in \mathcal{U}$ such that
$$z \geq J(t,x,\hat{u}) \qquad \text{and} \qquad X^{\hat{u}}_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.}$$
one has
$$v(t,x) \leq z,$$
that is the "$\leq$" inequality in (5.3.2) holds true.
In order to obtain the reverse inequality it is sufficient to observe that for any $\nu \in \mathcal{U}$ such that $X^\nu_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T]$ one has
$$J(t,x,\nu) \in \Big\{ z \geq 0 : \exists u \in \mathcal{U} \text{ such that } J(t,x,u) \leq z \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.} \Big\}.$$
Therefore
$$J(t,x,\nu) \geq \inf\Big\{ z \geq 0 : \exists u \in \mathcal{U} \text{ such that } J(t,x,u) \leq z \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.} \Big\}$$
and the result follows by the arbitrariness of ν. $\square$

The first important result of this section is contained in the theorem below. It concerns the link between the right side of (5.3.2), that is linked with what in literature is called a stochastic target problem with controlled expected loss (see the definition given in [61]), and a suitable stochastic target problem, where the target is asked to be reached almost surely. The proof, that we report here for completeness, is an adaptation to our case of the arguments presented in [61] and [58]. The idea at the basis of the result is that the problem with controlled-loss is equivalent to the corresponding stochastic target problem up to a martingale. This observation will result in an augmentation of the state and control space by adding a martingale component to the original dynamics.

Proposition 5.3.2. *Let assumptions* (H_b),(H_σ) *and* $(H_{\psi,\ell})$ *be satisfied. Then*

$$v(t,x) = \inf \Big\{ z \geq 0 : \exists (u,\alpha) \in \mathcal{U} \times \mathcal{A} \text{ such that} \\ \Big(Z^{\alpha,u}_{t,x,z}(T) \geq \psi(X^u_{t,x}(T)) \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \Big) \text{ a.s.} \Big\}$$

where $\mathcal{A}$ *denotes the set square-integrable* $\mathbb{R}^p$*-valued predictable processes and* $Z^{\alpha,u}_{t,x,z}(\cdot)$ *is the one-dimensional process defined by*

$$Z^{\alpha,u}_{t,x,z}(\cdot) = z - \int_t^{\cdot} \ell(s, X^u_{t,x}(s), u(s))ds + \int_t^{\cdot} \alpha_s^T d\mathcal{B}(s). \tag{5.3.3}$$

Proof. We will prove that for any $z \geq 0$ the following equivalence holds:

$$\exists u \in \mathcal{U} : \mathbb{E}\left[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds \right] \leq z \\ \text{and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.} \\ \Longleftrightarrow \exists (u,\alpha) \in \mathcal{U} \times \mathcal{A} : \Big(Z^{\alpha,u}_{t,x,z}(T) \geq \psi(X^u_{t,x}(T)) \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \Big) \text{ a.s.}$$

where $Z^{\alpha,u}_{t,x,z}$ is the process defined by (5.3.3).
The left implication is trivial and it just follows by taking the expectation in the right hand term and recalling that if $\alpha \in \mathcal{A}$, $\int_t^{\cdot} \alpha^T(s) d\mathcal{B}(s)$ is a martingale.
Furthermore, under our assumptions, $\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds \in L^2(\Omega, \mathbb{F}_T; \mathbb{R})$ for any $u \in \mathcal{U}$, where $L^2(\Omega, \mathbb{F}_T; \mathbb{R})$ denotes the set of the $\mathbb{R}$-valued $\mathbb{F}_T$-measurable random variables η such that $\mathbb{E}[|\eta|^2] < \infty$. As a consequence the Ito's representation theorem applies (see [172, Theorem 5.7, Chapter I] for instance), that is there exists a process $\hat{\alpha} \in \mathcal{A}$ such that

$$z \geq J(t,x,u) = \psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds - \int_t^T \hat{\alpha}_s^T d\mathcal{B}(s).$$

Hence, defining

$$Z^{\hat{\alpha},u}_{t,x,z}(\cdot) := z - \int_t^{\cdot} \ell(s, X^u_{t,x}(s), u(s))ds + \int_t^{\cdot} \hat{\alpha}_s^T d\mathcal{B}(s),$$

the other implication is obtained and the statement of the theorem follows by Lemma 5.3.1. $\square$

Remark 5.3.3. It can be useful to remark that the set $\mathcal{A}$ of controls that appears in Proposition 5.3.2 can be restricted to the set of squared integrable $\mathbb{R}^p$-valued process uniformly bounded in the $L^2_{\mathbb{F}}$-norm.
In fact since α is defined by the Ito's representation theorem, one has:

$$\begin{aligned}\int_t^T \alpha^T(s)d\mathcal{B}_s &= \psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds + \\ &\quad -\mathbb{E}\big[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds\big]\end{aligned}$$

for any $t \in [0,T], x \in \mathbb{R}^d$. Thanks to the Ito's isometry and the linear growth of ψ and ℓ (consequence of $(H_{\psi,\ell})$) one has

$$\begin{aligned}\|\alpha\|^2_{L^2_{\mathbb{F}}} =&\mathbb{E}\Big[\int_t^T |\alpha_s|^2 ds\Big] = \mathbb{E}\Big[\Big|\int_t^T \alpha^T(s)d\mathcal{B}(s)\Big|^2\Big] \\ =&\mathbb{E}\Big[\Big|\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds \\ &- \mathbb{E}\big[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds\big]\Big|^2\Big] \\ \le& C_1\mathbb{E}\Big[\Big(1 + |X^u_{t,x}(T)|^2 + \int_t^T |X^u_{t,x}(s)|^2 ds\Big)\Big]\end{aligned}$$

where C_1 is a constant depending on L_ψ, L_ℓ and T. We recall that under assumption (H_b) and (H_σ) there exists a constant C_2 such that

$$\mathbb{E}\Big[\sup_{t\le s\le T} |X^u_{t,x}(s)|^2\Big] \le C_2(1 + |x|^2)$$

(see Proposition 2.1.1, Chapter 2), then a bound on the $L^2_{\mathbb{F}}$-norm of α of the form

$$\|\alpha\|_{L^2_{\mathbb{F}}} \le C(1 + |x|)$$

can be derived.

In order to simplify the notation, in some case we will directly consider the process

$$Y^{u,\alpha}_{t,y}(\cdot) \equiv (X^u_{t,x}(\cdot), Z^{\alpha,u}_{t,x,z}(\cdot))$$

solution of the following SDE in the augmented state space $\mathbb{R}^{d+1}$:

$$(5.3.4)\qquad \begin{cases} dY(s) = \tilde{b}(s, X(s), u(s))ds + \tilde{\sigma}(s, X(s), u(s), \alpha(s))d\mathcal{B}(s) & s \in [t,T] \\ Y(t) = y \equiv (x, z), \end{cases}$$

with

$$\tilde{b}(s, X, u) := \begin{pmatrix} b(s, X, u) \\ -\ell(s, X, u) \end{pmatrix}, \qquad \tilde{\sigma}(s, X, u, \alpha) := \begin{pmatrix} \sigma(s, X, u) \\ \alpha^T \end{pmatrix}.$$

Let $\mathcal{T} \subseteq \mathbb{R}^{d+1}$, the target set, be defined by

$$\mathcal{T} := \Big\{y \equiv (x, z) \in \mathbb{R}^{d+1} : z \ge \psi(x)\Big\} \equiv \text{Epigraph}(\psi).$$

The constrained condition for the variable $Y(\cdot)$ becomes:

$$Y_{t,y}^{u,\alpha}(s) \in \mathcal{K} \times \mathbb{R}, \forall s \in [t,T] \quad \text{a.s.} \ .$$

By Proposition 5.3.2 follows immediately that, defined the backward reachable set

$$\mathcal{R}_t^{\mathcal{T},\mathcal{K}} := \Big\{ y \in \mathbb{R}^{d+1} : \exists (u,\alpha) \in \mathcal{U} \times \mathcal{A} \text{ such that} \\ Y_{t,y}^{u,\alpha}(T) \in \mathcal{T} \text{ and } Y_{t,y}^{u,\alpha}(s) \in \mathcal{K} \times \mathbb{R}, \forall s \in [t,T] \text{ a.s.} \Big\}, \tag{5.3.5}$$

one has

$$v(t,x) = \inf \Big\{ z \geq 0 : (x,z) \in \mathcal{R}_t^{\mathcal{T},\mathcal{K}} \Big\}. \tag{5.3.6}$$

The immediate consequence of the equality above is that the computation of the backward reachable set $\mathcal{R}_t^{\mathcal{T},\mathcal{K}}$ can be the starting point for the characterization of v. We already provided in in Chapter 3, Section 3.2, the main references (as [163, 162, 56]) for the study of stochastic target problems in the unconstrained case. The constrained case has been considered in [59]. In these works a characterization of the solution in terms of discontinuous viscosity solution of a non-linear second order partial differential equation (a particular HJB equation) is proved. The presence of state constraints complicates the analysis and compatibility assumptions between the dynamics and the set $\mathcal{K}$ have still to be required. One of our main interest is also to provide a way for the computation of v in a very general setting and this aspect seems not yet be taken into account in the existing literature on stochastic target problems.
For this reason, in analogy with what we did in Chapter 3, we propose here to solve the target problem by a level set approach.

5.4 The level set approach

Re-write the optimal control problem as a reachability problem has the main advantage that the state constraints can be managed without requiring any further assumption on the system. It is due to the fact that state constrained reachability problems, as (5.3.5), can be solved by using a level set method. This approach is based on the idea of Osher and Sethian [147] (see also the references given in Chapter 3, Section 3.3) of characterizing a set, the backward reachable set $\mathcal{R}_t^{\mathcal{T},\mathcal{K}}$ for us, as a level set of suitable continuous function. Adapting to our setting the ideas introduced in [122] and [47] for the deterministic case, it will be proved in this section that the set $\mathcal{R}_t^{\mathcal{T},\mathcal{K}}$ can be seen as a level set of the value function associated to a suitable *unconstrained* auxiliary optimal control problem. The arguments we propose here are close to those presented in Chapter 3. There are mainly two differences: the first one is the necessity to take into account the unbounded controls α and the second one is that we will privilege the use an integral penalization of the state constraints. As shown in Chapter 3, the fact of considering a "maximum" penalization in the stochastic case, makes necessary to add an auxiliary variable for managing the past history of the running maximum. In the context of this chapter, adding the variable z, we already augmented the dimension of the problem and, for this reason, it seems better to us to avoid a further increase of the dimension. Moreover, we saw that dealing with maximum costs makes the theoretical analysis more complicated so, to include unbounded controls in this framework could further complicate the study. Let us consider a function $g_{\mathcal{K}}$ such that:

$(H_{g_{\mathcal{K}}})$
$$g_{\mathcal{K}} : \mathbb{R}^d \to \mathbb{R} \text{ is a Lipschitz continuous fuction,}$$
$$g_{\mathcal{K}}(x) \geq 0 \quad \text{and} \quad g_{\mathcal{K}}(x) = 0 \Leftrightarrow x \in \mathcal{K}.$$

Remark 5.4.1. The requirements on the function $g_{\mathcal{K}}$ are the same of Chapter 3, Section 3.3. So if $\mathcal{K}$ is a nonempty and closed set it is sufficient to take $g_{\mathcal{K}}(x) = d^+_{\mathcal{K}}(x)$.

Let us also introduce the function $g_\psi : \mathbb{R}^{d+1} \to \mathbb{R}$ defined by

$$g_\psi(x, z) := \max\big(\psi(x) - z, 0\big).$$

We consider the following *unconstrained* optimal control problem

$$w(t, x, z) := \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\Big[g_\psi(X^u_{t,x}(T), Z^{\alpha,u}_{t,x,z}(T)) + \int_t^T g_{\mathcal{K}}(X^u_{t,x}(s))ds\Big] \tag{5.4.1}$$

The following assumption will be also considered:

(H_0) for any $(t, x, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$ there exists an optimal pair of controls $(\bar{u}, \bar{\alpha}) \in \mathcal{U} \times \mathcal{A}$ for the optimal control problem (5.4.1).

Remark 5.4.2. When the set of control values is compact, the condition

$$(b, \sigma\sigma^T, \ell)(t, x, U) \quad \text{is convex}$$

is sufficient to guarantee, in the case of a weak formulation for problems of the form (5.2.3) and in absence of state constraints, the existence of an optimal control (see Section 2.2 and the references therein). In [135] and [107] this results is extended to the case of unbounded controls. In these papers some coercivity conditions replace the compactness assumption. At the moment we don't have knowledge of existence results that apply to our optimal control problem (5.4.1). Until now, using the arguments in [172], we were only able to rigorously prove an existence result if b, ℓ and σ are linear functions with respect to the space and the control and under some convexity assumptions on $g_{\mathcal{K}}$ and ψ. The proof is given in the Section 5.9 at the end of the chapter.

Theorem 5.4.3. *Let us assume* (H_b),(H_σ),$(H_{\psi,\ell})$ *and* (H_0) *be satisfied. Then*

$$\mathcal{R}^{\mathcal{T},\mathcal{K}}_t = \Big\{(x, z) \in \mathbb{R}^{d+1} : w(t, x, z) = 0\Big\}$$

and

$$v(t, x) = \inf\Big\{z \geq 0 : w(t, x, z) = 0\Big\}.$$

Proof. The latter statement clearly follows by the first one and by (5.3.6).
If $(x, z) \in \mathcal{R}^{\mathcal{T},\mathcal{K}}_t$, by the definition of the backward reachable set, there exists a couple $(\bar{u}, \bar{\alpha}) \in \mathcal{U} \times \mathcal{A}$ such that

$$\Big(g_\psi(X^{\bar{u}}_{t,x}(T), Z^{\bar{\alpha},\bar{u}}_{t,x,z}(T)) = 0 \quad \text{and} \quad g_{\mathcal{K}}(X^{\bar{u}}_{t,x}(s)) = 0, \quad \forall s \in [t, T]\Big) \qquad \text{a.s..}$$

It means that $w(t,x,z)=0$, therefore "$\subseteq$" inclusion is proved.
Let us now assume that $w(t,x,z)=0$. Thanks to assumption (H_0), there exists $(\bar u,\bar\alpha)\in\mathcal{U}\times\mathcal{A}$ such that

$$\mathbb{E}\Big[g_{\psi}(X^{\bar u}_{t,x}(T),Z^{\bar\alpha,\bar u}_{t,x,z}(T))+\int_t^T g_{\mathcal{K}}(X^{\bar u}_{t,x}(s))ds\Big]=0.$$

Recalling that g_ψ and $g_{\mathcal{K}}$ are positive functions, it implies

$$g_{\psi}(X^{\bar u}_{t,x}(T),Z^{\bar\alpha,\bar u}_{t,z}(T))+\int_t^T g_{\mathcal{K}}(X^{\bar u}_{t,x}(s))ds=0 \qquad \text{a.s..}$$

Hence,

$$\Big((X^{\bar u}_{t,x}(T),Z^{\bar\alpha,\bar u}_{t,x,z}(T))\in\mathcal{T} \quad \text{and} \quad X^{\bar u}_{t,x}(s)\in\mathcal{K},\quad \forall s\in[t,T]\Big) \qquad \text{a.s.}$$

and the result is proved. □

5.5 The HJB equation

In this section we aim to characterize the auxiliary value function w as the unique solution, in the viscosity sense, of a suitable partial differential equation. As we will see, the main difficulties come from the unboundedness of the control α.
Let us start by proving some regularity properties of w:

Proposition 5.5.1. *Let assumptions (H_b),(H_σ), ($H_{\psi,\ell}$) and ($H_{g_{\mathcal{K}}}$) be satisfied. Then w is continuous with respect to t and Lipschitz continuous with respect to (x,z). Moreover it satisfies the following growth condition on $[0,T]\times\mathbb{R}^d\times[0,+\infty)$:*

$$0\le w(t,x,z)\le C(1+|x|).$$

Proof. Let us start by proving the Lipschitz continuity in the space variables.
One has

$$\begin{aligned}
&|w(t,x,z)-w(t,x',z')|\\
&\le \sup_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}}\Big\{\mathbb{E}\Big[\big|g_\psi(X^u_{t,x}(T),Z^{u,\alpha}_{t,x,z}(T))-g_\psi(X^u_{t,x'}(T),Z^{u,\alpha}_{t,x',z'}(T))\big|\Big]+\\
&\qquad+\mathbb{E}\Big[\int_t^T\big|g_{\mathcal{K}}(X^u_{t,x}(s))-g_{\mathcal{K}}(X^u_{t,x'}(s))\big|ds\Big]\Big\}\\
&\le C\sup_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}}\Big\{\mathbb{E}\Big[\big|X^u_{t,x}(T)-X^u_{t,x'}(T)\big|+\big|Z^{u,\alpha}_{t,x,z}(T)-Z^{u,\alpha}_{t,x',z'}(T)\big|+\\
&\qquad+\mathbb{E}\Big[\int_t^T\big|X^u_{t,x}(s)-X^u_{t,x'}(s)\big|ds\Big]\Big\}
\end{aligned}$$

where the constant $C\ge 0$ depends on the Lipschitz constants of ψ and $g_{\mathcal{K}}$. Recalling that under assumption (H_b) and (H_σ) the estimates of Proposition 2.1.1 hold for the diffusion process $X^u_{t,x}(\cdot)$ and observing that

$$\big|Z^{u,\alpha}_{t,x,z}(T)-Z^{u,\alpha}_{t,x',z'}(T)\big|\le|z-z'|+\int_t^T|\ell(s,X^u_{t,x}(s),u(s))-\ell(s,X^u_{t,x'}(s),u(s))|ds,$$

thanks to the Lipschitz continuity of ℓ the result is obtained.
Let us now prove the continuity in time. Let be $t \leq t' \leq T$. By the properties of the infimum one has that for any $\varepsilon > 0$ there exists a pair $(u_\varepsilon, \alpha_\varepsilon) \in \mathcal{U} \times \mathcal{A}$ such that

$$\begin{aligned} w(t,x,z) - w(t',x,z) \leq \mathbb{E}\Big[\big| g_\psi(X^{u_\varepsilon}_{t,x}(T), Z^{u_\varepsilon,\alpha_\varepsilon}_{t,x,z}(T)) - g_\psi(X^{u_\varepsilon}_{t',x}(T), Z^{u_\varepsilon,\alpha_\varepsilon}_{t',x,z}(T)) \big| \Big] \\ + \mathbb{E}\Big[\Big| \int_t^T g_\mathcal{K}(X^{u_\varepsilon}_{t,x}(s))ds - \int_{t'}^T g_\mathcal{K}(X^{u_\varepsilon}_{t',x}(s))ds \Big| \Big] + \varepsilon. \end{aligned}$$

Then thanks again to the Lipschitz continuity of g_ψ and $g_\mathcal{K}$ one has

$$\begin{aligned} & w(t',x,z) - w(t,x,z) \\ & < \mathbb{E}\Big[\big| g_\psi(X^{u_\varepsilon}_{t,x}(T), Z^{u_\varepsilon,\alpha_\varepsilon}_{t,x,z}(T)) - g_\psi(X^{u_\varepsilon}_{t',x}(T), Z^{u_\varepsilon,\alpha_\varepsilon}_{t',x,z}(T)) \big| \Big] + \\ & \quad + \mathbb{E}\Big[\Big| \int_t^T g_\mathcal{K}(X^{u_\varepsilon}_{t,x}(s))ds - \int_{t'}^T g_\mathcal{K}(X^{u_\varepsilon}_{t',x}(s))ds \Big| \Big] + \varepsilon \\ & \leq \mathbb{E}\Big[\big| g_\psi(X^{u_\varepsilon}_{t,x}(T), Z^{u_\varepsilon,\alpha_\varepsilon}_{t,x,z}(T)) - g_\psi(X^{u_\varepsilon}_{t',x}(T), Z^{u_\varepsilon,\alpha_\varepsilon}_{t',x,z}(T)) \big| \Big] + \\ & \quad + \mathbb{E}\Big[\int_{t'}^T |g_\mathcal{K}(X^{u_\varepsilon}_{t,x}(s)) - g_\mathcal{K}(X^{u_\varepsilon}_{t',x}(s)| ds + \int_t^{t'} |g_\mathcal{K}(X^{u_\varepsilon}_{t,x}(s))| ds \Big] + \varepsilon \\ & \leq C\mathbb{E}\Big[|X^{u_\varepsilon}_{t,x}(T) - X^{u_\varepsilon}_{t',x}(T)| M_g(t'-t) + \\ & \quad + \int_{t'}^T |X^{u_\varepsilon}_{t,x}(s) - X^{u_\varepsilon}_{t',x}(s)| ds + \big| Z^{u_\varepsilon,\alpha_\varepsilon}_{t,x,z}(T) - Z^{u_\varepsilon,\alpha_\varepsilon}_{t',x,z}(T) \big| \Big] + \varepsilon. \end{aligned}$$

For the terms depending only on $X^{u_\varepsilon}_{\cdot,x}(\cdot)$, thanks to the estimates of Proposition 2.1.1 we can obtain a $\frac{1}{2}$-Hölder regularity in time. So let us only consider the variable $Z^{u_\varepsilon,\alpha_\varepsilon}_{\cdot,x,z}(\cdot)$. By its definition one has

$$\begin{aligned} & \big| Z^{u_\varepsilon,\alpha_\varepsilon}_{t,x,z}(T) - Z^{u_\varepsilon,\alpha_\varepsilon}_{t',x,z}(T) \big| \\ & \leq \Big| \int_t^T \ell(s, X^\varepsilon_{t,x}(s), u_\varepsilon(s))ds - \int_{t'}^T \ell(s, X^{u_\varepsilon}_{t,x}(s), u_\varepsilon(s))ds \Big| + \Big| \int_t^{t'} (\alpha_\varepsilon)_s^T d\mathcal{B}(s) \Big|. \end{aligned}$$

Again for the first term in the right hand side of the inequality we can use Proposition 2.1.1 and obtaining $\frac{1}{2}$-Hölder continuity. On the other hand the continuity of the term depending on α_ε comes from the fact that

$$\lim_{t' \to t} \Big| \int_t^{t'} (\alpha_\varepsilon)_s^T d\mathcal{B}(s) \Big| = 0$$

since $\alpha_\varepsilon \in L^2_\mathbb{F}(0,T;\mathbb{R}^p)$. It follows that

$$\limsup_{t' \to t} \big(w(t',x,z) - w(t,x,z) \big) \leq 0$$

locally uniformly. By the same arguments

$$\liminf_{t' \to t} \big(w(t',x,z) - w(t,x,z) \big) \geq 0,$$

so the continuity is proved.
Concerning the growth condition, by the definition of w we have

$$w(t,x,z) \leq \inf_{(u,\alpha)\in\mathcal{U}\times\{0\}} \mathbb{E}\Big[\max\big(\psi(X^u_{t,x}(T)) - Z^{0,u}_{t,x,z}(T),0\big) + \int_t^T g_{\mathcal{K}}(X^u_{t,x}(s))ds\Big]. \tag{5.5.1}$$

In particular, if $z \geq 0$, thanks to the positivity of ℓ and ψ it leads to

$$w(t,x,z) \leq \inf_{u\in\mathcal{U}} \mathbb{E}\Big[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s,X^u_{t,x}(s),u(s)) + g_{\mathcal{K}}(X^u_{t,x}(s))ds\Big]$$

and the desired result is obtained thanks to the linear growth of ψ,ℓ and $g_{\mathcal{K}}$ and classical estimates for the process $X^u_{t,x}(\cdot)$ (see [172, Theorem 6.3, Chapter I], for instance). □

Another information that will be useful later concerns the value assumed by w for $z \leq 0$.

Lemma 5.5.2. *Under assumptions* (H_b),(H_σ),$(H_{\psi,\ell})$ *and* $(H_{g_{\mathcal{K}}})$, *for any* $t\in[0,T], x\in\mathbb{R}^d, z\leq 0$ *one has*

$$w(t,x,z) = w_0(t,x) - z,$$

where

$$w_0(t,x) := \inf_{u\in\mathcal{U}} \mathbb{E}\Big[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s,X^u_{t,x}(s),u(s)) + g_{\mathcal{K}}(X^u_{t,x}(s))ds\Big].$$

Proof. We can start observing that, thanks to the definition of the function g_ψ, for any $(t,x,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}$

$$\begin{aligned}
w(t,x,z) &= \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\Big[\max\big(\psi(X^u_{t,x}(T)) - Z^{\alpha,u}_{t,x,z}(T),0\big) + \int_t^T g_{\mathcal{K}}(X^u_{t,x}(s))ds\Big] \\
&\geq \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\Big[\psi(X^u_{t,x}(T)) - z + \int_t^T \alpha^T(s)d\mathcal{B}_s \\
&\qquad + \int_t^T \ell(s,X^u_{t,x}(s),u(s)) + g_{\mathcal{K}}(X^u_{t,x}(s))ds\Big] \\
&= w_0(t,x) - z,
\end{aligned}$$

using that $\int_t^T \alpha^T(s)d\mathcal{B}_s$ is a martingale. Hence the "$\geq$" inequality is satisfied for any $z\in\mathbb{R}$.
On the other hand we can use (5.5.1). If $z\leq 0$, thanks to the positivity of ℓ and ψ we have

$$\begin{aligned}
&\max\big(\psi(X^u_{t,x}(T)) - Z^{0,u}_{t,x,z}(T),0\big) \\
&= \max\big(\psi(X^u_{t,x}(T)) - z + \int_t^T \ell(s,X^u_{t,x}(s),u(s)),0\big) \\
&= \psi(X^u_{t,x}(T)) - z + \int_t^T \ell(s,X^u_{t,x}(s),u(s))
\end{aligned}$$

and we get

$$w(t,x,z) \leq w_0(t,x) - z.$$

□

By the standard dynamic programming approach the candidate Hamiltonian associated to the optimal control problem (5.4.1) is

$$H(t,x,q,Q) := \sup_{(u,\alpha)\in U\times\mathbb{R}^p} \left\{ -\tilde{b}(t,x,u)q - \frac{1}{2}Tr[\tilde{\sigma}\tilde{\sigma}^T(t,x,u,\alpha)Q] \right\} - g_{\mathcal{K}}(x)$$

where $q \in \mathbb{R}^{d+1}$ and $Q \in \mathcal{S}^{d+1}$. Recalling the definition of $\tilde{\sigma}$ we have

$$\tilde{\sigma}\tilde{\sigma}^T(t,x,u,\alpha) = \begin{pmatrix} \sigma\sigma^T(t,x,u) & \sigma(t,x,u)\alpha \\ \alpha^T\sigma^T(t,x,u) & \alpha^T\alpha \end{pmatrix}.$$

By the unboundedness of the control $\alpha \in \mathbb{R}^p$ follows the possible unboundedness of the Hamiltonian H and this may pose some problems for the definition of the concept of solution for the associated HJB equation. We refer to [150] for the application of the HJ techniques in the case of unbounded Hamiltonians.

Remark 5.5.3. In our case it is not possible to follow the approach in [150] for dealing with the unbounded Hamiltonian. In fact by continuity arguments we cannot find a continuous function $G : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d+1} \times \mathcal{S}^{d+1} \to \mathbb{R}$ such that

$$H(t,x,q,Q) < +\infty \Leftrightarrow G(t,x,q,Q) \leq 0.$$

The approach we will apply is based on the ideas presented in [64] and exploited also in [45]. In these works, thanks to an alternative formulation of the HJB equation satisfied by w, a numerical scheme is proposed without turning to any a priori restriction on the set of the values of α. We point out that similar "compactification" results have been obtained in [153, 154] for deterministic control systems and in [144, 145] for stochastic problems where the unbounded control acts only on the drift.
In what follows we denote by $\Lambda^+(A)$ the bigger eigenvalue of a given matrix A.

Lemma 5.5.4. *For any $t \in (0,T), (x,z) \in \mathbb{R}^{d+1}, q \in \mathbb{R}^{d+1}, r \in \mathbb{R}$ and $Q \in \mathcal{S}^{d+1}$, let us define $A, C \in \mathbb{R}$ and $B \in \mathbb{R}^p$ as follows:*

$$\begin{gathered} a := 2(-r - \tilde{b}(t,x,u)\cdot q - \tfrac{1}{2}Tr[\sigma\sigma^T(t,x,u)Q_{11}] - g_{\mathcal{K}}(x)), \\ B := (B_1,\dots,B_p)^T = \sigma^T(t,x,u)Q_{12}, \qquad c := Q_{22} \end{gathered}$$

where $Q_{11} \in \mathcal{S}^d, Q_{22} \in \mathbb{R}, Q_{12} = Q_{21}^T \in \mathbb{R}^{d\times 1}$ denotes the blocks of the symmetric matrix Q. If $\mathcal{H}^u \in \mathcal{S}^{p+1}$ is the matrix defined by blocks by

$$\begin{aligned} \mathcal{H}^u(t,x,r,q,Q) &:= \begin{pmatrix} a & -B^T \\ -B & -cI_p \end{pmatrix} \\ &= \begin{pmatrix} a & -B_1 & -B_2 & \dots & -B_p \\ -B_1 & -c & 0 & \dots & 0 \\ -B_2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -B_p & 0 & \dots & 0 & -c \end{pmatrix} \end{aligned}$$

the following assertions hold:

(i) $-r + H(t,x,q,Q) \leq 0 \Leftrightarrow \sup_{u\in U} (\Lambda^+(\mathcal{H}^u(t,x,r,q,Q))) \leq 0;$

(ii) $-r + H(t,x,q,Q) \le 0 \Rightarrow Q_{22} \ge 0$;

(iii) $-r + H(t,x,q,Q) = 0 \Rightarrow \sup_{u\in U} (\Lambda^+(\mathcal{H}^u(t,x,r,q,Q))) = 0$;

(iv) $\sup_{u\in U} (\Lambda^+(\mathcal{H}^u(t,x,r,q,Q))) < 0 \Rightarrow -r + H(t,x,q,Q) < 0$.

Proof. By the definition of $\tilde{\sigma}$ and by the fact that

$$Tr[\sigma\alpha Q_{21}] = \alpha^T\sigma^T Q_{21}^T = \alpha^T\sigma^T Q_{12},$$

one can easily verify that the Hamiltonian H takes the following form:

$$H(t,x,q,Q) = \sup_{\substack{(u,\alpha)\\ \in U\times\mathbb{R}^p}} \left\{ -\tilde{b}\cdot q - \frac{1}{2}Tr[\sigma\sigma^T Q_{11}] - \alpha^T\sigma^T Q_{12} - \frac{1}{2}\|\alpha\|^2 Q_{22} \right\} - g_{\mathcal{K}}(x).$$

In particular it follows that

$$\begin{array}{rc} H(t,x,q,Q) = +\infty & \text{if } Q_{22} < 0 \\ & \text{or } Q_{22} = 0 \text{ and } \sigma^T Q_{12} \neq 0 \\ H(t,x,q,Q) \in \mathbb{R} & \text{otherwise} \end{array}$$

and, as a consequence, (ii) holds.
Moreover the following equivalences can be obtained:

$$\begin{aligned}
& -r + H(t,x,q,Q) \le 0 \\
\Leftrightarrow\ & \sup_{\substack{(u,\alpha)\\ \in U\times\mathbb{R}^p}} \left\{ -r - \tilde{b}\cdot q - \frac{1}{2}Tr[\sigma\sigma^T Q_{11}] - g_{\mathcal{K}}(x) - \alpha^T\sigma^T Q_{12} - \frac{1}{2}\alpha^T\alpha Q_{22} \right\} \le 0 \\
\Leftrightarrow\ & \sup_{(u,\alpha)\in U\times\mathbb{R}^p} \left\{ a - 2\alpha^T B - c\|\alpha\|^2 \right\} \le 0 \\
\Leftrightarrow\ & \sup_{\substack{(u,\beta)\in U\times\mathbb{R}^{p+1},\\ \beta_1\neq 0}} \left\{ a - 2\sum_{i=1}^p \frac{\beta_{i+1}}{\beta_1} B_i - c\sum_{i=1}^p \frac{\beta_{i+1}^2}{\beta_1^2} \right\} \le 0 \\
\Leftrightarrow\ & \sup_{\substack{(u,\beta)\in U\times\mathbb{R}^{p+1},\\ \|\beta\|=1}} \left\{ \beta_1^2 a - 2\sum_{i=1}^p \beta_{i+1}\beta_1 B_i - c\sum_{i=1}^p \beta_{i+1}^2 \right\} \le 0 \\
\Leftrightarrow\ & \sup_{u\in U} \sup_{\substack{\beta\in\mathbb{R}^{p+1}\\ \|\beta\|=1}} \beta^T \mathcal{H}^u(t,x,r,q,Q)\beta \le 0 \\
\Leftrightarrow\ & \sup_{u\in U} \Lambda^+(\mathcal{H}^u(t,x,r,q,Q)) \le 0
\end{aligned}$$

and (i) is proved. In the same way point (iii) is obtained and finally (iv) follows. □

Remark 5.5.5. We point out that for any $\lambda > 0$

$$\operatorname{sign} \Lambda^+ \begin{pmatrix} a & -B^T \\ -B & -cI_p \end{pmatrix} = \operatorname{sign} \Lambda^+ \begin{pmatrix} a & -\lambda B^T \\ -\lambda B & -\lambda^2 cI_p \end{pmatrix}.$$

In fact (we denote $0 \equiv (0, \dots, 0)^T \in \mathbb{R}^p$)

$$
\begin{aligned}
&\Lambda^+ \begin{pmatrix} a & -\lambda B^T \\ -\lambda B & -\lambda^2 c I_p \end{pmatrix} \le 0 \\
&\Leftrightarrow \sup_{x \in \mathbb{R}^{p+1}, x \neq 0} \langle \begin{pmatrix} a & -\lambda B^T \\ -\lambda B & -\lambda^2 c I_p \end{pmatrix} x, x \rangle \le 0 \\
&\Leftrightarrow \sup_{x \in \mathbb{R}^{p+1}, x \neq 0} \langle \begin{pmatrix} 1 & 0^T \\ 0 & \lambda I_p \end{pmatrix} \begin{pmatrix} a & -B^T \\ -B & -c I_p \end{pmatrix} \begin{pmatrix} 1 & 0^T \\ 0 & \lambda I_p \end{pmatrix} x, x \rangle \le 0 \\
&\Leftrightarrow \sup_{x \in \mathbb{R}^{p+1}, x \neq 0} \langle \begin{pmatrix} 1 & 0^T \\ 0 & \lambda I_p \end{pmatrix} \begin{pmatrix} a & -B^T \\ -B & -c I_p \end{pmatrix} \begin{pmatrix} 1 & 0^T \\ 0 & \lambda I_p \end{pmatrix} x, \begin{pmatrix} 1 & 0^T \\ 0 & \lambda I_p \end{pmatrix} x \rangle \le 0.
\end{aligned}
$$

Defining the vector

$$y = \begin{pmatrix} 1 & 0^T \\ 0 & \lambda I_p \end{pmatrix} x$$

it is easy to verify that the last assertion is equivalent to

$$\sup_{y \in \mathbb{R}^{p+1}, y \neq 0} \langle \begin{pmatrix} a & -B^T \\ -B & -c I_p \end{pmatrix} y, y \rangle \le 0.$$

For convenience of the reader we report below the explicit definition of $\mathcal{H}^u$:

$$
\begin{aligned}
&\mathcal{H}^u(t, x, r, q, Q) \\
&= \begin{pmatrix} -r - (b, \ell)(t, x, u) \cdot q - \frac{1}{2} Tr[\sigma\sigma^T(t, x, u) Q_{11}] - g_{\mathcal{K}}(x) & -\frac{1}{2} Q_{21} \sigma(t, x, u) \\ -\frac{1}{2} \sigma^T(t, x, u) Q_{12} & -\frac{1}{2} Q_{22} I_p \end{pmatrix}.
\end{aligned}
$$

Theorem 5.5.6. *w is a viscosity solution of the following equation:*

$$
(5.5.2) \qquad \begin{cases} \sup_{u \in U} \left(\Lambda^+ (\mathcal{H}^u(t, x, \partial_t w, Dw, D^2 w)) \right) = 0 & [0, T) \times \mathbb{R}^d \times (0, +\infty) \\ w(t, x, 0) = w_0(t, x) & [0, T) \times \mathbb{R}^d \\ w(T, x, z) = \max(\psi(x) - z, 0) & \mathbb{R}^d \times [0, +\infty) \end{cases}
$$

Proof. The boundary condition for $t = T$ and $z = 0$ are satisfied thanks to the definition of w and Lemma 5.5.2.
Let be $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^{d+1})$ such that

$$(w - \varphi)(\bar{t}, \bar{x}, \bar{z}) = \max_{[0,T) \times \mathbb{R}^{d+1}} (w - \varphi)$$

for $(\bar{t}, \bar{x}, \bar{z}) \in [0, T) \times \mathbb{R}^d \times (0, +\infty)$. By the DPP one can obtain that

$$-\partial_t \varphi(\bar{t}, \bar{x}, \bar{z}) + H(\bar{t}, \bar{x}, D\varphi, D^2\varphi) \le 0,$$

therefore by Lemma 5.5.4 follows

$$\sup_{u \in U} \Lambda^+ (\mathcal{H}^u(\bar{t}, \bar{x}, \partial_t \varphi, D\varphi, D^2\varphi)) \le 0$$

and the sub-solution property is proved.

Let be $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^{d+1})$ such that

$$(w - \varphi)(\bar{t}, \bar{x}, \bar{z}) = \min_{[0,T) \times \mathbb{R}^{d+1}} (w - \varphi).$$

Let us define the set

$$\mathcal{M}(\varphi) := \Big\{(t,x,z) : \sup_{u\in U}\big(\Lambda^+(\mathcal{H}^u(t,x,\partial_t\varphi, D\varphi, D^2\varphi))\big) < 0\Big\}.$$

If by contradiction $(\bar t,\bar x,\bar z)\in\mathcal{M}(\varphi)$, since $\mathcal{M}(\varphi)$ is an open set, there exists $\eta>0$ such that $S_\eta(\bar t,\bar x,\bar z) := [0\vee \bar t-\eta, \bar t+\eta]\times\overline{B((\bar x,\bar z),\eta)}\subset\mathcal{M}(\varphi)$. Applying the same techniques exploited in Lemma 3.1 in [150] it is possible to prove that

$$\inf_{\partial_P(S_\eta(\bar t,\bar x,\bar z))}(w-\varphi) = \min_{S_\eta(\bar t,\bar x,\bar z)}(w-\varphi)$$

where $\partial_P(S_\eta(\bar t,\bar x,\bar z)) := \big([0\vee\bar t-\eta,\bar t+\eta]\times\partial B((\bar x,\bar z),\eta)\big)\cup\big(\{\bar t+\eta\}\times\overline{B((\bar x,\bar z),\eta)}\big)$ it is the forward parabolic boundary of S_η. But since $(\bar t,\bar x,\bar z)$ is a strict minimizer, for η small enough the contradiction is obtained since $(\bar t,\bar x,\bar z)\notin\partial_p(S_\eta(\bar t,\bar x,\bar z))$. We can conclude that $(\bar t,\bar x,\bar z)\notin\mathcal{M}(\varphi)$ and w is a viscosity super-solution of (5.5.2). □

We point out that problem (5.5.2) is equivalent to

$$\begin{cases} \sup\limits_{\substack{u\in U,\xi\in\mathbb{R}^{p+1}\\ \|\xi\|=1}}\Big\{\xi^T\mathcal{H}^u(t,x,\partial_t w, Dw, D^2w)\xi\Big\} = 0 & [0,T)\times\mathbb{R}^d\times(0,+\infty)\\ w(t,x,0) = w_0(t,x) & [0,T)\times\mathbb{R}^d\\ w(T,x,z) = \max(\psi(x)-z,0) & \mathbb{R}^d\times[0,+\infty) \end{cases}$$

5.6 Comparison Principle

This section is devoted to the proof of a strong comparison principle between upper semi-continuous (USC) viscosity sub-solution and lower semi-continuous (LSC) super-solution of equation (5.5.2).
For this proof we follow the main lines of the proof presented in [64], extended here to the higher dimension.
In virtue of Remark 5.5.5, choosing

$$\lambda(z) := \max(1,z) > 0,$$

the equation considered is equivalent to the following (the dependence on (t,x,u) of b, σ and ℓ is omitted for space reasons):
(5.6.1)

$$\sup_{u\in U}\Lambda^+\begin{pmatrix} -\partial_t w - b\cdot D_x w + \ell\,\partial_z w - \frac{1}{2}Tr[\sigma\sigma^T D^2_{xx}w] - g_{\mathcal{K}}(x) & -\frac{1}{2}\lambda(z)D^T_{xz}w\,\sigma\\ -\frac{1}{2}\lambda(z)\sigma^T D_{xz}w & -\frac{1}{2}\lambda^2(z)\partial_{zz}w I_n\end{pmatrix} = 0$$

Theorem 5.6.1. *Let assumptions* (H_b),(H_σ),$(H_{\psi,\ell})$ *and* $(H_{g_{\mathcal{K}}})$ *be satisfied and let* $\underline{w}\in USC([0,T]\times\mathbb{R}^d\times[0,+\infty))$ *and* $\overline{w}\in LSC([0,T]\times\mathbb{R}^d\times[0,+\infty))$ *be respectively a viscosity sub- and super-solution of* (5.6.1). *Let us assume that* $\overline{w}$ *and* $\underline{w}$ *satisfy the following growth condition*

$$\underline{w}(t,x,z)\le C(1+|x|)\qquad \overline{w}(t,x,z)\ge C(1+|x|).$$

Then if

$$\underline{w}(T,x,z)\le\max(\psi(x)-z,0)\le\overline{w}(T,x,z)$$

and

$$\underline{w}(t,x,0)\le w_0(t,x)\le\overline{w}(t,x,0)$$

one has $\underline{w}(t,x,z)\le\overline{w}(t,x,z)$ *for all* $(t,x,z)\in[0,T]\times\mathbb{R}^d\times[0,+\infty)$.

Proof. We start proving that there exists a strict viscosity sub-solution of (5.6.1). Let us introduce the function ζ defined by

$$\zeta(t,z) := -(T-t) - \ln(1+z).$$

One has that $\zeta \in C^\infty([0,T]\times[0,+\infty))$ for all $(t,z)\in[0,T]\times[0,+\infty)$

$$\zeta(t,z) \le 0$$

and

$$\mathcal{H}_0^u(t,x,\partial_t\zeta,D\zeta,D^2\zeta) := \begin{pmatrix} -\partial_t\zeta - bD_x\zeta + \ell\,\partial_z\zeta - \frac{1}{2}Tr[\sigma\sigma^T D^2_{xx}\zeta] & -\frac{1}{2}\lambda(z)D^T_{xz}\zeta\,\sigma \\ -\frac{1}{2}\lambda(z)\sigma^T D_{xz}\zeta & -\frac{1}{2}\lambda^2(z)\partial_{zz}\zeta\, I_n \end{pmatrix}$$

$$= \begin{pmatrix} -1-\ell(t,x,u)\frac{1}{1+z} & 0 \quad \dots \quad 0 \\ 0 & \\ \vdots & -\frac{1}{2}\lambda^2(z)\frac{1}{(1+z)^2}I_n \\ 0 & \end{pmatrix}.$$

It follows immediately by the positivity of the function ℓ that $-1-\ell(t,x,u)/(1+z)\le -1$. Moreover using the fact that

$$\frac{1}{4} \le \frac{\max(1,z^2)}{(1+z)^2} \le \frac{1}{2}$$

we have

$$\sup_{u\in U} \Lambda^+(\mathcal{H}_0^u(t,x,z,\partial_t\zeta,D\zeta,D^2\zeta)) \le -\frac{1}{4}. \tag{5.6.2}$$

Let $\underline{w}_\eta$ be defined by

$$\underline{w}_\eta(t,x,z) := \underline{w}(t,x,z) + \eta\zeta(t,z).$$

We prove that for any $\eta>0$, $\underline{w}_\eta$ is a strict viscosity sub-solution of (5.6.1) in $[0,T]\times\mathbb{R}\times[0,+\infty)$. The boundary conditions are of course satisfied thanks to the non positivity of ζ and we have

$$\underline{w}_\eta(T,x,z) \le \psi(x) - z, \qquad \underline{w}_\eta(t,x,0) \le w_0(t,x).$$

Let us now consider a test function $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^d\times[0,+\infty))$ such that

$$(\underline{w}_\eta - \varphi)(t_0,x_0,z_0) = \max(\underline{w}_\eta - \varphi)$$

for $(t_0,x_0,z_0)\in[0,T)\times\mathbb{R}^d\times(0,+\infty)$. From the definition of $\underline{w}_\eta$, we have

$$\big(\underline{w} - (-\eta\zeta+\varphi)\big)(t_0,x_0,z_0) = \max\big(\underline{w} - (-\eta\zeta+\varphi)\big).$$

Therefore we can consider as a test function for $\underline{w}$

$$\psi := -\eta\zeta + \varphi.$$

By using the sub-linearity property of the operator Λ^+, the following inequalities hold:

$$\begin{aligned}
&\sup_{u\in U}\Lambda^+(\mathcal{H}^u(t_0,x_0,\partial_t\varphi,D\varphi,D^2\varphi)) \\
&= \sup_{u\in U}\Lambda^+\big(\mathcal{H}^u(t_0,x_0,\partial_t\psi,D\psi,D^2\psi) + \eta\,\mathcal{H}_0^u(t_0,x_0,\partial_t\zeta,D\zeta,D^2\zeta)\big) \\
&\le \sup_{u\in U}\Lambda^+(\mathcal{H}^u(t_0,x_0,z_0,\psi_t\psi,D\psi,D^2\psi)) + \eta\sup_{u\in U}\Lambda^+(\mathcal{H}_0^u(t_0,x_0,\partial_t\zeta,D\zeta,D^2\zeta)) \\
&\le -\frac{\eta}{4},
\end{aligned}$$

where for the last inequality we have used (5.6.2) and the fact that $\underline{w}$ is a sub-solution.
Let us define

$$\Phi_{\delta,\eta,\rho}(t,x,z) := \underline{w}_\eta(t,x,z) - \overline{w}(t,x,z) - 2\delta e^{-\rho t}(1+|x|^2+z).$$

Let $(\hat{t},\hat{x},\hat{z}) := (\hat{t}_{\delta,\eta,\rho},\hat{x}_{\delta,\eta,\rho},\hat{z}_{\delta,\eta,\rho})$ be a maximum point for $\Phi_{\delta,\eta,\rho}$ (this maximum point exists because of the linear growth condition satisfied by $\underline{w}_\eta - \overline{w}$).
Let us take into account the following three scenarios:

a) there exists a sequence $\{(\delta_k,\eta_k,\rho_k)_{k\geq 0}\}$ such that $\hat{t}=T$;

b) there exists a sequence $\{(\delta_k,\eta_k,\rho_k)_{k\geq 0}\}$ such that $\hat{z}=0$;

c) there is a sequence $\{(\delta_k,\eta_k,\rho_k)_{k\geq 0}\}$ such that $\Phi_{\delta_k,\eta_k,\rho_k}(\hat{t},\hat{x},\hat{z})\leq 0$.

If case a) occurs, for any (t,x,z) one has

$$\begin{aligned}
\underline{w}_{\eta_k}(t,x,z) - \overline{w}(t,x,z) &= \Phi_{\delta_k,\eta_k,\rho_k}(t,x,z) + 2\delta_k e^{-\rho_k t}(1+|x|^2+z)\\
&\leq \Phi_{\delta_k,\eta_k,\rho_k}(\hat{t},\hat{x},\hat{z}) + 2\delta_k(1+|x|^2+z)\\
&\leq \underline{w}(T,\hat{x},\hat{z}) - \overline{w}(T,\hat{x},\hat{z}) + 2\delta_k(1+|x|^2+z)\\
&\leq 2\delta_k(1+|x|^2+z).
\end{aligned}$$

Analogously in case b) one has

$$\begin{aligned}
\underline{w}_{\eta_k}(t,x,z) - \overline{w}(t,x,z) &\leq \underline{w}(\hat{t},\hat{x},0) - \overline{w}(\hat{t},\hat{x},0) + 2\delta_k(1+|x|^2+z)\\
&\leq 2\delta_k(1+|x|^2+z).
\end{aligned}$$

In case c) we have

$$\Phi_{\delta_k,\eta_k,\rho_k}(t,x,z)\leq 0$$

for any (t,x,z) and then again

$$\underline{w}_{\eta_k}(t,x,z) - \overline{w}(t,x,z) \leq 2\delta_k(1+|x|^2+z).$$

Hence in cases a), b) and c) we get

$$\underline{w}(t,x,z) + \eta_k\zeta(t,x,z) - \overline{w}(t,x,z) \leq 2\delta_k(1+|x|^2+z), \qquad \forall (t,x,z)$$

and in the limit $\delta_k,\eta_k\to 0$

$$(\underline{w}-\overline{w})(t,x,z)\leq 0$$

that gives the desired comparison result. For this reason in what follows we can assume that δ and η are small enough, such that, there exists $\gamma>0$ such that the maximum point $(\hat{t},\hat{x},\hat{z})$ of $\Phi_{\delta,\eta,\rho}$ satisfies, for any $\rho\geq 0$: $\hat{t}<T-\gamma$, $\hat{z}>\gamma$ and

$$\Phi_{\delta,\eta,\rho}(\hat{t},\hat{x},\hat{z})>0. \tag{5.6.3}$$

We aim to show a contradiction. From now on $\delta,\eta,\hat{t},\hat{x},\hat{z}$ are fixed.
The next passage is standard in order to prove comparison principles for viscosity solutions and consists in the doubling of variables.
We consider

$$\begin{aligned}
&\Phi_\varepsilon(t,x,x',z,z') :=\\
&\underline{w}_\eta(t,x,z) - \overline{w}(t,x',z') - \delta e^{-\rho t}(1+|x|^2+z) - \delta e^{-\rho t}(1+|x'|^2+z')\\
&-\frac{1}{2\varepsilon}(|x-x'|^2+|z-z'|^2) - \frac{1}{2}|t-\hat{t}|^2 - \frac{1}{4}(|x-\hat{x}|^4+|z-\hat{z}|^4).
\end{aligned}$$

Let $(\bar t,\bar x,\bar x',\bar z,\bar z') := (\bar t_\varepsilon,\bar x_\varepsilon,\bar x'_\varepsilon,\bar z_\varepsilon,\bar z'_\varepsilon)$ be a maximum point for Φ_ε.
By standard arguments (see [84, Proposition 3.7] for instance) we can prove that for $\varepsilon\to 0$ one has

$$|\bar x-\bar x'|\to 0,\quad |\bar z-\bar z'|\to 0$$

and

$$\bar t\to\hat t,\quad \bar x,\bar x'\to\hat x,\quad \bar z,\bar z'\to\hat z,\quad \frac{|\bar x-\bar x'|^2}{\varepsilon}\to 0,\quad \frac{|\bar z-\bar z'|^2}{\varepsilon}\to 0.$$

Let us define

$$\begin{aligned}
f_{\delta,\rho}(t,x,z) &:= \delta e^{-\rho t}(1+|x|^2+z)+\frac14|t-\hat t|^2+\frac14(|x-\hat x|^4+|z-\hat z|^4),\\
\hat f_{\delta,\rho}(t,x,z) &:= \delta e^{-\rho t}(1+|x|^2+z)\\
\varphi(x,x',z,z') &:= \frac{1}{2\varepsilon}(|x-x'|^2+|z-z'|^2)
\end{aligned}$$

so that we can write

$$\begin{aligned}
&\Phi_\varepsilon(t,x,x',z,z')\\
&:= \big(\underline{w}_\eta(t,x,z)-f_{\delta,\rho}(t,x,z)\big)-\big(\overline{w}(t,x',z')+\hat f_{\delta,\rho}(t,x',z')\big)-\varphi(x,x',z,z').
\end{aligned}$$

Applying the Crandall-Ishii Lemma (see Lemma 2.4.7, Chapter 2) we can find $r,r'\in\mathbb{R}$ and two symmetric matrices X and X' such that

$$\begin{aligned}
&r+r'=\partial_t\varphi(\bar x,\bar x',\bar z,\bar z')=0\\
&(r+\partial_t f_{\delta,\rho}, D_{(x,z)}(\varphi+f_{\delta,\rho}), X+D^2_{(x,z)}f_{\delta,\rho})\in\overline{\mathcal{P}}^{1,2,+}\underline{w}_\eta(\bar t,\bar x,\bar z)\\
&(-r'-\partial_t\hat f_{\delta,\rho}, -D_{(x',z')}(\varphi+\hat f_{\delta,\rho}), -X'-D^2_{(x',z')}\hat f_{\delta,\rho})\in\overline{\mathcal{P}}^{1,2,-}\overline{w}(\bar t,\bar x',\bar z')
\end{aligned}$$

and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\le\begin{pmatrix} X & 0\\ 0 & X'\end{pmatrix}\le\frac{3}{\varepsilon}\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}\tag{5.6.4}$$

(where $\overline{\mathcal{P}}^{1,2,\pm}$ denote the closures of the parabolic semijets, see Definition 2.4.6, Chapter 2).
From the definition of viscosity sub- and super-solution :

$$\begin{aligned}
&\sup_{u\in U}\Lambda^+(\mathcal{H}^u(\bar t,\bar x,r+\partial_t f_{\delta,\rho},D_{(x,z)}(\varphi+f_{\delta,\rho}),X+D^2_{(x,z)}f_{\delta,\rho}))\le-\frac{\eta}{4}\\
&\sup_{u\in U}\Lambda^+(\mathcal{H}^u(\bar t,\bar x',-r'-\partial_t\hat f_{\delta,\rho},-D_{(x',z')}(\varphi+\hat f_\delta),-X'-D^2_{(x',z')}\hat f_\delta))\ge 0.
\end{aligned}$$

Then using the following notation

$$\begin{aligned}
(q_1,q_2)^T &:= D_{(x,z)}(\varphi+f_{\delta,\rho})\quad Q:=D^2_{(x,z)}f_{\delta,\rho}\\
(q_1',q_2')^T &:= D_{(x',z')}(\varphi+\hat f_{\delta,\rho})\quad Q':=D^2_{(x',z')}\hat f_{\delta,\rho}
\end{aligned}$$

one has

$$\begin{aligned}
&\sup_{u\in U}\Lambda^+(\mathcal{H}^u(\bar t,\bar x',-r'-\partial_t\hat f_{\delta,\rho},-q',-X'-Q'))\\
&\qquad-\sup_{u\in U}\Lambda^+(\mathcal{H}^u(\bar t,\bar x,r+\partial_t f_{\delta,\rho},q,X+Q))\ge\frac{\eta}{4}.
\end{aligned}\tag{5.6.5}$$

We are going to estimate from above the left hand term. Let us define the matrices $A, A' \in \mathcal{S}^{p+1}$:

$$A := 2\begin{pmatrix} -r - \frac{1}{2}\partial_t f_{\delta,\rho} - b(\bar{t},\bar{x},u)q_1 + \ell(\bar{t},\bar{x},u)q_2 - g_{\mathcal{K}}(\bar{x}) & 0 \\ 0 & 0 \end{pmatrix}$$

$$A' := 2\begin{pmatrix} r' + \frac{1}{2}\partial_t \hat{f}_{\delta,\rho} + b(\bar{t},\bar{x}',u)q'_1 - \ell(\bar{t},\bar{x}',u)q'_2 - g_{\mathcal{K}}(\bar{x}') & 0 \\ 0 & 0 \end{pmatrix}.$$

We observe that

$$\begin{aligned}
&\mathcal{H}^u(\bar{t},\bar{x}', -r' - \partial_t \hat{f}_{\delta,\rho}, -q', -X' - Q') \\
&= A' + \begin{pmatrix} \frac{1}{2}\partial_t \hat{f}_{\delta,\rho} + Tr[\sigma\sigma^T(\bar{t},\bar{x}',u)(X'_{11}+Q'_{11})] & \lambda(\bar{z}')(X'_{12}+Q'_{12})^T\sigma(\bar{t},\bar{x}',u) \\ \lambda(\bar{z}')\sigma^T(\bar{t},\bar{x}',u)(X'_{12}+Q'_{12}) & \lambda^2(\bar{z}')(X'_{22}+Q'_{22})I_p \end{pmatrix} \\
&= A' + \widetilde{X}' + \widetilde{Q}'
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{H}^u(\bar{t},\bar{x}, r + \partial_t f_{\delta,\rho}, q, X + Q) \\
&= A + \begin{pmatrix} -\frac{1}{2}\partial_t f_{\delta,\rho} + Tr[\sigma\sigma^T(\bar{t},\bar{x},u)(X_{11}+Q_{11})] & \lambda(\bar{z})(X_{12}+Q_{12})^T\sigma(\bar{t},\bar{x},u) \\ \lambda(\bar{z})\sigma^T(\bar{t},\bar{x},u)(X_{12}+Q_{12}) & \lambda^2(\bar{z})(X_{22}+Q_{22})I_p \end{pmatrix} \\
&= A - \widetilde{X} - \widetilde{Q}
\end{aligned}$$

where we used the following definitions:

$$\widetilde{X} := \begin{pmatrix} Tr[\sigma\sigma^T(\bar{t},\bar{x},u)X_{11}] & \lambda(\bar{z})X_{12}^T\sigma(\bar{t},\bar{x},u) \\ \lambda(\bar{z})\sigma^T(\bar{t},\bar{x},u)X_{12} & \lambda^2(\bar{z})X_{22}I_p \end{pmatrix}$$

($\widetilde{X}'$ is the same expression for the matrix X' at point $(\bar{t},\bar{x}',\bar{z}')$) and

$$\widetilde{Q} := \begin{pmatrix} -\frac{1}{2}\partial_t f_{\delta,\rho} + Tr[\sigma\sigma^T(\bar{t},\bar{x},u)Q_{11}] & \lambda(\bar{z})Q_{12}^T\sigma(\bar{t},\bar{x},u) \\ \lambda(\bar{z})\sigma^T(\bar{t},\bar{x},u)Q_{12} & \lambda^2(\bar{z})Q_{22}I_p \end{pmatrix},$$

$$\widetilde{Q}' := \begin{pmatrix} \frac{1}{2}\partial_t \hat{f}_{\delta,\rho} + Tr[\sigma\sigma^T(\bar{t},\bar{x},u)Q'_{11}] & \lambda(\bar{z})(Q'_{12})^T\sigma(\bar{t},\bar{x},u) \\ \lambda(\bar{z})\sigma^T(\bar{t},\bar{x},u)Q'_{12} & \lambda^2(\bar{z})Q'_{22}I_p \end{pmatrix}.$$

Therefore, using again the sub-linearity of the operator Λ^+ we have

$$\begin{aligned}
&\sup_{u\in U}\Lambda^+(\mathcal{H}^u(\bar{t},\bar{x}', -r' - \partial_t \hat{f}_{\delta,\rho}, -p', -X' - Q')) \\
&\qquad - \sup_{u\in U}\Lambda^+(\mathcal{H}^u(\bar{t},\bar{x}, r + \partial_t f_{\delta,\rho}, p, X + Q)) \\
&\le \sup_{u\in U}\left\{\Lambda^+(A' + \widetilde{X}' + \widetilde{Q}') - \Lambda^+(A - \widetilde{X} - \widetilde{Q})\right\} \\
&\le \sup_{u\in U}\left\{\underbrace{\Lambda^+(A' - A)}_{(I)} + \underbrace{\Lambda^+(\widetilde{X} + \widetilde{X}')}_{(II)} + \underbrace{\Lambda^+(\widetilde{Q} + \widetilde{Q}')}_{(III)}\right\}
\end{aligned}$$

-Estimation for (I):
A direct computation shows that

$$
\begin{aligned}
\partial_t f_{\delta,\rho} &= \frac{1}{2}(\bar t - \hat t) - \delta\rho e^{-\rho \bar t}(1+|\bar x|^2 + \bar z) \\
\partial_t \hat f_{\delta,\rho} &= -\delta\rho e^{-\rho \bar t}(1+|\bar x'|^2 + \bar z') \\
q &= \begin{pmatrix} 2\delta e^{-\rho\bar t}\bar x + (\bar x - \hat x)|\bar x - \hat x|^2 + \frac{1}{\varepsilon}(\bar x - \bar x') \\ \delta e^{-\rho \bar t} + (\bar z - \hat z)^3 + \frac{1}{\varepsilon}(\bar z - \bar z') \end{pmatrix} \\
q' &= \begin{pmatrix} 2\delta e^{-\rho\bar t}\bar x' - \frac{1}{\varepsilon}(\bar x - \bar x') \\ \delta e^{-\rho \bar t} - \frac{1}{\varepsilon}(\bar z - \bar z') \end{pmatrix}
\end{aligned}
$$

so

$$
\begin{aligned}
&\Lambda^+(A'-A) \\
&= 2\max\Big(\frac{1}{2}\partial_t f_{\delta,\rho} + \frac{1}{2}\partial_t \hat f_{\delta,\rho} + b(\bar t, \bar x', u)q_1' \\
&\qquad - \ell(\bar t, \bar x', u)q_2' - g_{\mathcal K}(\bar x') + b(\bar t, \bar x, u)q_1 - \ell(\bar t, \bar x, u)q_2 + g_{\mathcal K}(\bar x)\ ,\ 0\Big) \\
&\le \max\Big(\frac{1}{2}|\bar t - \hat t| - \delta\rho e^{-\rho\bar t}(1+|\bar x|^2+\bar z) - \delta\rho e^{-\rho\bar t}(1+|\bar x'|^2+\bar z') \\
&\qquad + C\Big(\frac{|\bar x - \bar x'|^2}{\varepsilon} + \frac{|\bar z - \bar z'||\bar x - \bar x'|}{\varepsilon} + |\bar x - \bar x'|\big(1+\delta e^{-\rho\bar t} + \delta e^{-\rho \bar t}|\bar x'|\big) \\
&\qquad + (1+|\bar x|)(\delta e^{-\rho\bar t} + \delta e^{-\rho\bar t}|\bar x| + \delta e^{-\rho\bar t}|\bar x'| + |\bar x - \hat x|^3 + |\bar z - \hat z|^3)\Big)\ ,\ 0\Big)
\end{aligned}
$$

where C is a constant depending only on $L_b, L_\ell, L_{g_{\mathcal K}}$ (we used also the fact that being b and σ Lipschitz continuous, they have a linear growth). Taking the limit for $\varepsilon \to 0$ we have

$$
\limsup_{\varepsilon\to 0} \Lambda^+(A'-A) \le \max\Big(-\delta\rho e^{-\rho\hat t}(1+|\hat x|^2+\hat z) + C\delta e^{-\rho\hat t}(1+|\hat x|^2)\ ,\ 0\Big). \tag{5.6.6}
$$

Choosing ρ large enough so that the first argument in the right hand side of (5.6.6) is negative, we obtain

$$
\limsup_{\varepsilon\to 0} \Lambda^+(A'-A) \le 0.
$$

-Estimation for (II):

Let us start defining the matrices $\Sigma, \Sigma' \in \mathbb{R}^{(p+1)\times(d+1)}$:

$$
\Sigma := \begin{pmatrix} \sigma^T(\bar t, \bar x, u) & (0\dots 0)^T \\ 0\dots 0 & \lambda(\bar z) \end{pmatrix} \qquad \Sigma' := \begin{pmatrix} \sigma^T(\bar t, \bar x', u) & (0\dots 0)^T \\ 0\dots 0 & \lambda(\bar z') \end{pmatrix}.
$$

Let ξ be an arbitrary vector in $\mathbb{R}^{p+1}$. Multiplying inequality (5.6.4) by $\xi^T(\Sigma \quad \Sigma')$ on the left side and by $(\Sigma \quad \Sigma')^T\xi$ on the right side we obtain

$$
\xi^T\left(\Sigma \quad \Sigma'\right)\begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix}\begin{pmatrix} \Sigma^T \\ \Sigma'^T \end{pmatrix}\xi \le \frac{3}{\varepsilon}\,\xi^T\left(\Sigma \quad \Sigma'\right)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\begin{pmatrix} \Sigma^T \\ \Sigma'^T \end{pmatrix}\xi.
$$

that gives:

$$
\xi^T(\Sigma' X' \Sigma'^T + \Sigma X \Sigma^T)\xi \le \frac{3}{\varepsilon}\,\xi^T(\Sigma - \Sigma')(\Sigma^T - \Sigma'^T)\xi \le \frac{3}{\varepsilon}\|\Sigma - \Sigma^T\|_F^2\|\xi\|^2
$$

where $\|\cdot\|_F$ denotes the Frobenius matrix norm. In particular choosing ξ of the form

$$\xi \equiv (0\dots 0\ \underbrace{\beta_0}_{k\text{-th}}\ 0\dots 0\ \beta_k)$$

a straightforward calculation gives

$$\begin{aligned}
&\beta_0^2\big(\sigma^T X_{11}\sigma(\bar t,\bar x,u)+\sigma^T X'_{11}\sigma(\bar t,\bar x',u)\big)_{kk}+\beta_k^2\big(\lambda^2(\bar z)X_{22}+\lambda^2(\bar z')X'_{22}\big)\\
&+2\beta_0\beta_k\big(\lambda(\bar z)\sigma^T(\bar t,\bar x,u)X_{12}+\lambda(\bar z')\sigma^T(\bar t,\bar x',u)X'_{12}\big)_k\\
&\le \frac{3}{\varepsilon}\|\Sigma-\Sigma'\|_F^2(\beta_0^2+\beta_k^2), \qquad \forall k=1,\dots,n.
\end{aligned}$$

It follows that for any $\beta\in\mathbb{R}^{p+1}$

$$\begin{aligned}
&\beta^T(\widetilde X+\widetilde X')\beta\\
&=\sum_{k=1}^n\Big(\beta_0^2\big(\sigma^T X_{11}\sigma(\bar t,\bar x,u)+\sigma^T X'_{11}\sigma(\bar t,\bar x',u)\big)_{kk}\\
&\quad+2\beta_0\beta_k\big(\lambda(\bar z)\sigma^T(\bar t,\bar x,u)X_{12}+\lambda(\bar z')\sigma^T(\bar t,\bar x',u)X'_{12}\big)_k+\beta_k^2\big(\lambda^2(\bar z)X_{22}+\lambda^2(\bar z')X'_{22}\big)\Big)\\
&\le \frac{3}{\varepsilon}\|\Sigma-\Sigma'\|_F^2\|\beta\|^2.
\end{aligned}$$

It is now sufficient to observe that, thanks to the definition of Σ and Σ' and the Lipschitz continuity of σ

$$\frac{3}{\varepsilon}\|\Sigma-\Sigma'\|_F^2\le 3\Big(L_\sigma^2\frac{|\bar x-\bar x'|^2}{\varepsilon}+\frac{|\bar z-\bar z'|^2}{\varepsilon}\Big)\xrightarrow{\varepsilon\to 0}0$$

and we get

$$\limsup_{\varepsilon\to 0}\Lambda^+(\widetilde X+\widetilde X')\le 0.$$

-Estimation for (III):
A direct calculation shows that

$$\begin{aligned}
Q&=\begin{pmatrix}(2\delta e^{-\rho\bar t}+|\bar x-\hat x|^2)I_d+(\bar x-\hat x)(\bar x-\hat x)^T & (0\dots 0)^T\\ 0\dots 0 & 3|\bar z-\hat z|^2\end{pmatrix}\\
Q'&=\begin{pmatrix}2\delta e^{-\rho\bar t}I_d & (0\dots 0)^T\\ 0\dots 0 & 0\end{pmatrix}
\end{aligned}$$

so

$$\begin{aligned}
&\Lambda^+(\widetilde Q'+\widetilde Q)\\
&=\max\Big(\frac12\partial_t f_{\delta,\rho}+\frac12\partial_t\hat f_{\delta,\rho}+Tr[\sigma\sigma^T(\bar t,\bar x,u)Q_{11}]+Tr[\sigma\sigma^T(\bar t,\bar x',\bar z')Q'_{11}]\ ,\\
&\qquad\lambda^2(\bar z)Q_{22}+\lambda^2(\bar z')Q'_{22}\Big)\\
&=\max\Big(\frac14(\bar t-\hat t)-\frac12\delta\rho e^{-\rho\bar t}(1+|\bar x|^2+\bar z)-\frac12\delta\rho e^{-\rho\bar t}(1+|\bar x'|^2+\bar z')\\
&\qquad+\sum_{i,j}\big(2\delta e^{-\rho\bar t}+|\bar x-\hat x|^2+(\bar x-\hat x)^2_{ij})\sigma^2_{ij}(\bar t,\bar x,u)+2\delta e^{-\rho\bar t}\sigma^2_{ij}(\bar t,\bar x',u)\big)\ ,\\
&\qquad 3\lambda^2(\bar z)|\bar z-\hat z|^2\Big).
\end{aligned}$$

Recalling that σ has linear growth and that $|\bar{x}|^2, \bar{z}$ are bounded ($\bar{z}$ is also non negative) as $\varepsilon \to 0$ we have

$$\begin{aligned}
&\Lambda^+(\widetilde{Q}' + \widetilde{Q}) \\
&\leq \max\Bigg(\frac{1}{4}|\bar{t}-\hat{t}| - \frac{1}{2}\delta\rho e^{-\rho\bar{t}}(1+|\bar{x}|^2+\bar{z}) - \frac{1}{2}\delta\rho e^{-\rho\bar{t}}(1+|\bar{x}'|^2+\bar{z}') \\
&\qquad + C_1\delta e^{-\rho\bar{t}}(1+|\bar{x}|^2+|\bar{x}'|^2) + C(1+|\bar{x}|^2)|\bar{x}-\hat{x}|^2 \ , \ 3\lambda^2(\bar{z})|\bar{z}-\hat{z}|^2\Bigg) \\
&\leq \max\Bigg(\frac{1}{4}|\bar{t}-\hat{t}| + \delta e^{-\rho\bar{t}}(1+|\bar{x}|^2+|\bar{x}'|^2)(C_1 - \frac{\rho}{2}) + C_2|\bar{x}-\hat{x}|^2 \ , \ C_3|\bar{z}-\hat{z}|^2\Bigg).
\end{aligned}$$

Taking $\rho/2 \geq C_1$, in the limit $\varepsilon \to 0$ one obtains

$$\limsup_{\varepsilon\to 0} \Lambda^+(\widetilde{Q}' + \widetilde{Q}) \leq \limsup_{\varepsilon\to 0} \ \max\left(\frac{1}{4}|\bar{t}-\hat{t}| + C_2|\bar{x}-\hat{x}|^2 \ , \ C_3|\bar{z}-\hat{z}|^2\right) = 0.$$

Hence, using estimations $(I), (II)$ and (III), from inequality (5.6.5) we finally get, as $\varepsilon \to 0$:

$$\frac{\eta}{4} \leq 0$$

that leads to the desired contradiction. □

5.7 The uncontrolled case

In this section we aim to determine the cost associated with a diffusion process under state constraints. Let us denote by $X_{t,x}(\cdot)$ the strong solution of the following stochastic differential equation

$$\begin{cases} dX(s) = b(X(s))ds + \sigma(X(s))d\mathcal{B}(s) & s \in [t,T] \\ X(t) = x. \end{cases}$$

Given a cost $\psi \geq 0$, the value function we aim to compute is

$$v(t,x) := \begin{cases} \mathbb{E}\Big[\psi(X_{t,x}(T))\Big] & \text{if } X_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \quad \text{a.s.} \\ +\infty & \text{otherwise.} \end{cases}$$

In the unconstrained case $\mathcal{K} = \mathbb{R}^d$ it is well known that the Faynman-Kac formula gives a characterization of v as the unique viscosity solution of the linear PDE

$$\begin{cases} -\partial_t v - Dv \cdot b(x) - \frac{1}{2}Tr[\sigma\sigma^T(x)D^2v] = 0 & t \in [0,T), x \in \mathbb{R}^d, \\ v(T,x) = \psi(x) & x \in \mathbb{R}^d. \end{cases}$$

When state constraints are taken into account, that is if $\mathcal{K} \subset \mathbb{R}^d$, such a characterization may become a very delicate issue since in absence of further requirements on the coefficients b and σ of the equation the value function is not guaranteed to be continuous (see the discussion in the introduction of the chapter). In this section we will see how the method we presented applies to this special (uncontrolled) case.

Let us consider the following optimal control problem

$$w(t,x,z) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\max(\psi(X_{t,x}(T)) - Z^\alpha_{t,z}(T), 0) + \int_t^T d^+_{\mathcal{K}}(X_{t,x}(s))ds\right]$$

with

$$Z^{\alpha}_{t,z}(\cdot) := z + \int_t^T \alpha^T(s)d\mathcal{B}(s).$$

Proposition 5.7.1. *Let assumptions* (H_b),(H_σ),(H_ψ) *and* $(H_\mathcal{K})$ *be satisfied. Let us also assume that* $\psi \geq 0$. *One has*

$$v(t,x) = \inf\Big\{z \geq 0 : w(t,x,z) = 0\Big\}.$$

Proof. The result follows by Theorem 5.4.3 since assumption (H_0) is satisfied (see Section 5.9 below). □

Once established this result we can proceed with the PDE characterization of w. Let us assume $p = 1$. Thanks to Theorem 5.5.6 and 5.6.1 we can state that w is the unique viscosity solution of the following generalized HJB equation

$$\begin{cases} \sup\limits_{\xi_1^2+\xi_2^2=1} \Big\{\xi_1^2(-\partial_t w - Dw\cdot b(x) - \frac{1}{2}Tr[\sigma\sigma^T(x)D_x^2 w] - d^+_\mathcal{K}(x) & \\ \qquad +\xi_1\xi_2\sigma^T(x)D_{xz}w - \xi_2^2\partial_{zz}w\Big\} = 0 & t\in[0,T), x\in\mathbb{R}^d, z\in(0,+\infty) \\ w(t,x,0) = w_0(t,x) & t\in[0,T), x\in\mathbb{R}^d \\ w(T,x,z) = \max(\psi(x)-z,0) & x\in\mathbb{R}^d, z\in[0,+\infty) \end{cases}$$

with

$$w_0(t,x) := \mathbb{E}\Big[\psi(X_{t,x}(T)) + \int_t^T d^+_\mathcal{K}(X_{t,x}(s))ds\Big].$$

We point out that even if we started from an uncontrolled problem, that usually leads to a linear PDE, the necessity of adding the auxiliary controlled variable z makes the final equation nonlinear.

5.8 An application to the electricity market

In this section we will discuss the application of our approach in a specific case coming from the electricity market.

We consider the electricity demand for the period $[t,T]$ governed by the following stochastic differential equation in $\mathbb{R}$:

$$dX(s) = b(X(s))ds + \sigma(X(s))d\mathcal{B}(s) \qquad s\in[t,T], \qquad X(t) = x.$$

The volume $Y(\cdot)$ of electricity remaining at every instant in the reserve, controlled by the debit u, is given by

$$dY(s) = (\beta(s) - u(s))ds \qquad s\in[t,T], \qquad Y(t) = y.$$

We will assume that the debit, that is our control, takes values in a compact set $U := [0,u_{max}]$. The value $Y(\cdot)$ is required to remain bounded by a maximum value Y_{max} along all the interval $[t,T]$. This is expressed by the following constraint on the state:

$$Y(s) \in [0, Y_{\max}] =: \mathcal{K}, \qquad \forall s\in[0,T].$$

Aim of the controller is to minimize the cost of production of electricity necessary to supply the demand $X(\cdot)$. At any time s the cost of production is given by

$$\ell(X(s), u(s)) := C(X(s) - ku(s)),$$

for some function $C : \mathbb{R} \to [0, +\infty)$ of the form $C(\xi) = \mu \max(\xi, 0)$ $(\mu > 0)$ and some constant $k > 0$ (in other words if $X(s) \leq ku(s)$ we can cover the request of energy with a debit $u(s)$ and the cost is $\ell(X(s), u(s)) = 0$, otherwise, if $X(s) > ku(s)$, we cannot satisfy the request with the reserve and this will lead to a positive cost). This results in the following state constrained optimal control problem

$$v(t,(x,y)) := \inf_{u\in\mathcal{U}} \left\{ \mathbb{E}\left[\int_t^T C(X_{t,x}(s) - ku(s))ds \right] : Y^u_{t,y}(s) \in [0, Y_{max}] \right\}.$$

Applying our approach we introduce the auxiliary unconstrained optimal control problem

$$w(t,(x,y),z) := \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\left[\max\Big(- Z^{u,\alpha}_{t,x,z}(T), 0\Big) + \int_t^T d^+_{[0,Y_{max}]}(Y^u_{t,y}(s))ds \right]$$

with

$$Z^{u,\alpha}_{t,x,z}(\cdot) = z - \int_t^{\cdot} \ell(X_{t,x}(s), u(s))ds + \int_t^{\cdot} \alpha^T(s)d\mathcal{B}(s).$$

In this case the existence of an optimal couple $(\bar{u}, \bar{\alpha})$ is guaranteed thanks to the linearity of the dynamics, then we have

$$v(t,(x,y)) = \inf \Big\{ z \geq 0 : w(t,(x,y),z) = 0 \Big\}.$$

Therefore, in virtue of Theorems 5.5.6 and 5.6.1, the optimal value $v(t,(x,y))$ is completely determined by the solution of the following problem:

$$\begin{cases} \sup\limits_{\substack{u\in U,\\ \xi\in B(0,1)}} \Big\{ \xi_1^2(-\partial_t w - b(x)\partial_x w - (\beta(t) - u)\partial_y w + \ell(x,u)\partial_z w - \frac{1}{2}\sigma^2(x)\partial_{xx} w - d^+_{[0,Y_{max}]}) \\ \qquad + \xi_1\xi_2\sigma(x)\partial_{xz} w - \xi_2^2 \partial_{zz} w \Big\} = 0 & t \in [0,T), (x,y) \in \mathbb{R}^2, z \in (0, +\infty) \\ w(t,(x,y),0) = w_0(t,(x,y)) & t \in [0,T), (x,y) \in \mathbb{R}^2 \\ w(T,(x,y),z) = -z & (x,y) \in \mathbb{R}^2, z \in [0, +\infty) \end{cases}$$

with

$$w_0(t,(x,y)) = \inf_{u\in\mathcal{U}} \mathbb{E}\left[\int_t^T \Big(\ell(X_{t,x}(s), u(s)) + d^+_{[0,Y_{max}]}(Y^u_{t,y}(s)) \Big) ds \right]$$

We remand to [45] and [92, Section 9.4] for discussions concerning the numerical approximation of this kind of equation.

5.9 Appendix: A result of existence of optimal controls for linear stochastic differential equations

In this section we aim to discuss the existence of an optimal control, required by assumption (H_0) in Section 5.4, for the optimal control problem (5.4.1). The optimal control we consider is

$$w(t,x,z) = \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} J(t,x,z,u,\alpha) \tag{5.9.1}$$

associated with the cost

$$J(t,x,z,u,\alpha) := \mathbb{E}\Big[g_{\psi}(X^u_{t,x}(T), Z^{\alpha,u}_{t,x,z}(T)) + \int_t^T g_{\mathcal{K}}(X^u_{t,x}(s))ds \Big].$$

As already pointed out in Remark 5.4.2, the main difficulties arise because of the unboundedness of the control α and the unique result we are able to prove rigorously concerns the linear case. The proof reported below is strongly based on the arguments in [172, Theorem 5.2, Chapter II] (see also Section 3.9 in Chapter 3 where the same techniques are applied to problems with maximum cost).

The following assumptions will be considered:

$$(E_1') \qquad \begin{cases} (i) \quad \mathcal{B} \text{ is a one-dimensional Brownian motion, that is } p=1;\\ (ii) \quad b,\sigma,\ell : [0,T]\times\mathbb{R}^d\times U \to \mathbb{R}^d \text{ are given by:}\\ \qquad b(t,x,u) = A(t)x + B(t)u,\\ \qquad \sigma(t,x,u) = C(t)x + D(t)u\\ \qquad \ell(t,x,u) = E(t)x + F(t)u\\ \text{where } A,B,C,D,E \text{ and } F \text{ are } L^\infty \text{continuous functions}\\ \text{with value in matrix spaces of suitable sizes.} \end{cases}$$

Let us also consider the following convexity assumptions:

(E_2) $\qquad U \subset \mathbb{R}^m$ is a convex and compact set;

(E_3') $\qquad \psi$ and $g_{\mathcal{K}}$ are Lipscitz and convex functions.

Remark 5.9.1. Also in this case (see Remark 3.9.1) if $g_{\mathcal{K}} = d^+_{\mathcal{K}}$, in order to satisfy assumption (E_3'), it is sufficient that $\mathcal{K}$ is a closed and convex set. Moreover we can observe that the convexity of ψ automatically implies the convexity of $g_{\psi}(x,z) = \max(\psi(x) - z, 0)$.

Theorem 5.9.2. *Let assumptions* (E_1'),(E_2) *and* (E_3') *be satisfied. Then for any* $t \in [0,T], (x,z) \in \mathbb{R}^{d+1}$ *such that* (5.9.1) *is finite, there exists an optimal control* $(\bar{u},\bar{\alpha}) \in \mathcal{U}\times\mathcal{A}$.

Proof. Let $(u_j,\alpha_j)_{j\geq 1} \in \mathcal{U}\times\mathcal{A}$ a sequence of minimizing controls, that is such that

$$\lim_{j\to+\infty} J(t,x,z,u_j,\alpha_j) = w(t,x,z).$$

Thanks to the compactness of U and the uniform bound on the $L^2_{\mathbb{F}}$-norm that can be obtained on the control α, see Remark 5.3.3, we can extract from (u_j,α_j) a subsequence (still indexed with j) such that

$$(u_j,\alpha_j) \to (\bar{u},\bar{\alpha}) \qquad \text{weakly in } L^2_{\mathbb{F}}\text{-norm}.$$

As a consequence of the Mazur's lemma, there exists a linear combination of (u_j,α_j)

$$(\tilde{u}_j,\tilde{\alpha}_j) := \sum_{i\geq 1}\lambda_{ij}(u_{i+j},\alpha_{i+j}), \qquad \lambda_{ij}\geq 0, \sum_{i\geq 1}\lambda_{ij} = 1$$

strongly convergent to the same limit, that is

$$(\tilde{u}_j,\tilde{\alpha}_j) \to (\bar{u},\bar{\alpha}) \qquad \text{strongly in } L^2_{\mathbb{F}}\text{-norm}.$$

One can observe that $(\tilde{u}_j, \tilde{\alpha}_j)$ and $(\bar{u}, \bar{\alpha})$ are still elements of $\mathcal{U} \times \mathcal{A}$ thanks to the convexity and closure of U.
Thanks to assumption (E_1'), it is easy to verify that

$$X_{t,x}^{\tilde{u}_j}(\cdot) = \sum_{i\geq 1} \lambda_{ij} X_{t,x}^{u_{i+j}}(\cdot), \qquad Z_{t,x,z}^{\tilde{u}_j,\tilde{\alpha}_j}(\cdot) = \sum_{i\geq 1} \lambda_{ij} Z_{t,x,z}^{u_{i+j},\alpha_{i+j}}(\cdot)$$

and

$$\left(X_{t,x}^{\tilde{u}_j}, Z_{t,x,z}^{\tilde{u}_j,\tilde{\alpha}_j}\right) \longrightarrow \left(X_{t,x}^{\bar{u}}, Z_{t,x,z}^{\bar{u},\bar{\alpha}}\right) \qquad \text{strongly in } L_{\mathbb{F}}^{\infty}\text{-norm.}$$

Then one has for any $\varepsilon > 0$ there exists $\bar{j} = \bar{j}(\varepsilon)$ such that for any $j > \bar{j}(\varepsilon)$

$$\begin{aligned}
&J(t, x, z, \bar{u}, \bar{\alpha}) \\
&= \mathbb{E}\left[g_{\psi}(X_{t,x}^{\tilde{u}_j}(T), Z_{t,x,z}^{\tilde{u}_j,\tilde{\alpha}_j}(T)) + \int_t^T g_{\mathcal{K}}(Z_{t,x}^{\tilde{u}_j}(s))ds\right] + \varepsilon \\
&\leq \mathbb{E}\left[\sum_{i\geq 1} \lambda_{ij} g_{\psi}(X_{t,x}^{u_{i+j}}(T), Z_{t,x,z}^{u_{i+j},\alpha_{i+j}}(T)) + \int_t^T g_{\mathcal{K}}(Z_{t,x}^{u_{i+j}}(s))ds\right] + \varepsilon \\
&\leq \sum_{i\geq 1} \lambda_{ij} \mathbb{E}\left[g_{\psi}(X_{t,x}^{u_{i+j}}(T), Z_{t,x,z}^{u_{i+j},\alpha_{i+j}}(T)) + \int_t^T g_{\mathcal{K}}(Z_{t,x}^{u_{i+j}}(s))ds\right] + \varepsilon \\
&= w(t, x, z) + 2\varepsilon
\end{aligned}$$

that means $(\bar{u}, \bar{\alpha})$ is optimal for (5.9.1). $\square$

Chapter 6

Ergodic state constrained stochastic optimal control problems

6.1 Introduction

This chapter is devoted to the study of ergodic optimal controls problems in presence of state constraints. Given a discount factor $\lambda > 0$, let $v_\lambda : \mathbb{R}^d \to \mathbb{R}$ be the value function associated to the following infinite horizon optimal control problem

$$v_\lambda(x) = \inf_{u \in \mathcal{U}} \mathbb{E}\Big[\int_0^\infty e^{-\lambda t} \ell(X_x^u(t), u(t))dt\Big]. \tag{6.1.1}$$

We are are going to study the limit of λv_λ as λ goes to 0. The classical ergodic optimal control problem concerns the long time behavior of the average $\frac{1}{T} v_T$ where v_T is given by

$$v_T(x) := \inf_{u \in \mathcal{U}} \mathbb{E}\Big[\int_0^T \ell(X_x^u(t), u(t))dt + \psi(X_x^u(T))\Big].$$

By classical Abelian-Tauberian theorems it is possible to prove, under appropriate assumptions, that the convergence for $T \to +\infty$ of $\frac{1}{T} v_T$ is equivalent to the convergence of the discounted problem λv_λ as $\lambda \to 0$. In virtue of this observation, in this chapter we will focus only on the infinite horizon problem (6.1.1). For any $\lambda > 0$ fixed, it is well known (see Chapter 2) that, applying the dynamic programming techniques, the value function v_λ can be characterized as the solution of the following HJB equation

$$\lambda v + H(x, Dv, D^2 v) = 0, \tag{6.1.2}$$

with

$$H(x, q, Q) := \sup_{u \in U} \Big\{ - b(x, u) \cdot q - \frac{1}{2} Tr[\sigma\sigma^T(x, u)Q] - \ell(x, u)\Big\}.$$

Dealing with ergodic problems, the typical result we look for is to prove the uniform convergence of λv_λ to a certain constant Λ and the convergence of $v_\lambda - v_\lambda(0)$ to a continuous function χ. Under suitable stability assumptions on the HJB equation, passing to the limit in (6.1.2), the function χ will result a viscosity solution of

$$\Lambda + H(x, D\chi, D^2\chi) = 0. \tag{6.1.3}$$

Equation (6.1.3) is called the *cell problem*. Related issues are the uniqueness of the constant Λ and its characterization in terms of an invariant measure for the stochastic system.
These questions have been addressed by several authors in the last years. We refer, among the others, to [77, 7, 8, 76, 42, 3] for the case of first order equations and to [9, 157, 156, 75, 151, 53, 54, 38, 52, 66, 55, 6] for the stochastic case. In what follows we aim to study this kind of problems taking into account the presence of state constraints. Dealing with state constraints the following three scenarios are usually taken into account:

- The trajectories are constrained on a torus: it is expressed by an assumption of periodicity on the coefficients of the dynamics;
- The trajectories are reflected on the boundary of the constraint: it brings to an equation with Neumann boundary conditions (see [134]);
- The optimal control problem is restricted to the trajectories that satisfy the state constraints.

In the deterministic case an overview of this three cases is presented in [7] and [8]. In the stochastic setting we refer to [9] for a clear presentation of the periodic case. The reflected case is studied in [157, 28, 43] (see also the references therein) and in [66, 52] for constraints represented by a polyhedral cone. The state constrained case with singular boundary conditions is studied in [131]. No results seem to be available at the moment if v_λ is a solution of the state constrained HJB equation (see [117, 160, 161])

$$(6.1.4) \qquad \begin{cases} \lambda v + H(x, Dv, D^2v) = 0 & \text{in } int(\mathcal{K}) \\ \lambda v + H(x, Dv, D^2v) \geq 0 & \text{in } \partial\mathcal{K}. \end{cases}$$

The study presented in this chapter aim to be a first step in this direction in the particular case of invariant (possibly unbounded) domains and under some assumptions of asymptotic flatness of the diffusion process.

The chapter in organized as follows: we introduce our setting, including the hypothesis of invariance of the domain, in Section 6.2. A study of the optimal control problem for λ fixed is proposed in Section 6.3. The asymptotic flatness assumption is presented in Section 6.4 where the ergodic problem is considered.

6.2 Setting

Let be given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and a p-dimensional Brownian motion $\mathcal{B}(\cdot)$.
The following system of controlled SDE's in $\mathbb{R}^d$ is considered

$$(6.2.1) \qquad \begin{cases} dX(t) = b(X(t), u(t))dt + \sigma(X(t), u(t))d\mathcal{B}(t) & t \geq 0, x \in \mathbb{R}^d \\ X(0) = x, \end{cases}$$

where $u \in \mathcal{U}$, the set of progressively measurable processes with values in a compact set $U \subset \mathbb{R}^m$ $(m \geq 1)$. We will work under the regularity assumptions (H_b) and (H_σ) for the coefficients of the SDE. Under these assumptions, for any choice of $u \in \mathcal{U}$ we will denote by $X_x^u(\cdot)$ the unique strong solution of (6.2.1) starting at point $x \in \mathbb{R}^d$.

Let us consider a cost function $\ell : \mathbb{R}^d \times U \to \mathbb{R}$ such that:

$$(H'_\ell) \qquad \begin{cases} (i) & \ell(\cdot,\cdot) \text{ is continuous on } \mathbb{R}^d \times U; \\ (ii) & \text{there is } \omega_\ell \text{ modulus of continuity s.t. } \forall x, y \in \mathbb{R}^d, u \in U: \\ & \quad |\ell(x,u) - \ell(y,u)| \le \omega_\ell(|x-y|); \\ (iii) & \exists M_\ell \ge 0 \text{ such that } \forall x \in \mathbb{R}^d, u \in U: \\ & \quad |\ell(x,u)| \le M_\ell. \end{cases}$$

Thanks to the boundedness of ℓ, without loss of generality we can assume that the modulus ω_ℓ is concave and sublinear.
Let $\lambda > 0$ be a given discount factor and $\mathcal{K}$ a non empty closed subset of $\mathbb{R}^d$, not necessarily bounded. We consider in this chapter the ergodic problem associated with the cost functional

$$J_\lambda(x, \alpha) := \mathbb{E}\left[\int_0^\infty e^{-\lambda t} \ell(X_x^u(t), u(t)) dt\right] \tag{6.2.2}$$

when the trajectory $X_x^u(\cdot)$ is required to satisfy almost surely (a.s.) some state constraints

$$X_x^u(t) \in \mathcal{K}, \qquad \forall t \ge 0.$$

The optimal control problem associated with the cost in (6.2.2) is what is called in literature an infinite horizon optimal control problem (see the presentation given in Chapter 2, Section 2.1). In this chapter we aim to study the behavior for $\lambda \to 0^+$ of the value function v_λ defined by

$$v_\lambda(x) := \inf_{u \in \mathcal{U}} \left\{ J_\lambda(x, u) : X_x^u(t) \in \mathcal{K}, \forall t \ge 0 \text{ a.s. } \right\}. \tag{6.2.3}$$

As extensively pointed out in Chapter 5, in the study of optimal control problems the presence of state constraints generates additional difficulties because of the loss of regularity of the value function on the boundary. A wide literature is nowadays available concerning the study of this kind of problems and the conditions that guarantees a characterization of v_λ as the unique viscosity solution of the state constrained HJB equation (see for instance [117, 35, 62] and the other references given in Chapter 5). For the study of the ergodic problem we will consider a simplified setting assuming that all the trajectories that start in $\mathcal{K}$ remain almost surely in $\mathcal{K}$ for any time $t \ge 0$. In other words we will assume that the set $\mathcal{K}$ satisfies the following assumptions:

$$(H''_\mathcal{K}) \qquad \begin{cases} (i) & \mathcal{K} \subseteq \mathbb{R}^d \text{ is a nonempty and closed set;} \\ (ii) & \mathcal{K} \text{ is invariant for (6.2.1) : for any } x \in \mathcal{K} \text{ and } u \in \mathcal{U} \\ & \quad X_x^u(t) \in \mathcal{K}, \forall t \ge 0 \text{ a.s. .} \end{cases}$$

This framework strongly simplifies the study of problem (6.2.3) that, under assumption ($H''_\mathcal{K}$), can be treated as an unconstrained optimal control problem from the moment that the trajectories "naturally" satisfy the constraints. By the way the solution of the ergodic problem remains nontrivial and it represents, for the techniques proposed, a first step for a future study.

Remark 6.2.1. It has been proved in [25] that a necessary and sufficient condition for the invariance of a closed domain $\mathcal{K}$ is that for any $x \in \partial\mathcal{K}$

$$b(x,u)p + \frac{1}{2}Tr[\sigma\sigma^T(x,u)Y] \geq 0, \qquad \forall u \in U, (p,Y) \in \mathcal{N}^2_{\mathcal{K}}(x) \tag{6.2.4}$$

where $\mathcal{N}^2_{\mathcal{K}}(x)$ is the second order normal cone at point (see Definition 2.5.2 and Theorem 2.5.3 in Chapter 2). Let us assume that the domain $\mathcal{K}$ is the closure of an open set and that the boundary $\partial\mathcal{K}$ is at least twice continuously differentiable. In this case using the characterization of the second order normal cone given in [84, Remark 2.7] one has that (6.2.4) is satisfied as soon as

$$\mathbf{n}(x)\sigma(x,u) = 0 \text{ and } b(x,u)Dd_{\mathcal{K}}(x) + \frac{1}{2}Tr[\sigma\sigma^T(x,u)D^2d_{\mathcal{K}}(x)] \geq 0, \quad \forall u \in U$$

where $\mathbf{n}$ is the exterior normal vector and $d_{\mathcal{K}}$ denotes the signed distance function to $\partial\mathcal{K}$, that is

$$d_{\mathcal{K}}(x) := \begin{cases} d^+_{\mathcal{K}}(x) & \text{if } \; x \in \mathbb{R}^d \setminus \mathcal{K} \\ -d^+_{\mathcal{K}}(x) & \text{if } \; x \in \mathcal{K}. \end{cases}$$

Remark 6.2.2. Another approach for the studying of the invariance property of $\mathcal{K}$ is based on the level set approach (see Chapter 3). Let us introduce a function $g_{\mathcal{K}} : \mathbb{R}^d \to \mathbb{R}$ satisfying assumption ($H_{g_{\mathcal{K}}}$) in Chapter 3 and let us consider the following value function (for some $\mu > 0$)

$$w_{viab}(x) := \sup_{u\in\mathcal{U}} \mathbb{E}\Big[\max_{t\geq 0} g_{\mathcal{K}}(X^u_x(t))e^{-\mu t}\Big]$$

(resp.

$$\widetilde{w}_{viab}(x) := \sup_{u\in\mathcal{U}} \mathbb{E}\Big[\int_0^{+\infty} g_{\mathcal{K}}(X^u_x(t))e^{-\mu t}dt\Big]).$$

It can be proved (see the proof of Proposition 3.3.2 in Chapter 3) that

$$\mathcal{K} \text{ is invariant } \Leftrightarrow w_{viab}(x) = 0 \quad (\text{resp. } \widetilde{w}_{viab}(x) = 0) \quad \forall x \in \mathcal{K}.$$

We point out that if assumption ($H''_{\mathcal{K}}$) is satisfied one has

$$\mathcal{U}_{\mathcal{K}}(x) := \Big\{u \in \mathcal{U} : X^u_x(t) \in \mathcal{K}, \forall t \geq 0 \text{ a.s.}\Big\} = \mathcal{U}, \qquad \forall x \in \mathcal{K}$$

and we can simply write the optimal control problem (6.2.3) as

$$v_\lambda(x) := \inf_{u\in\mathcal{U}} J_\lambda(x,u).$$

6.3 The HJB equation for λ fixed

In this section we state some result concerning the optimal control problem (6.2.3). Let us start with the following result concerning the regularity of v_λ.

Proposition 6.3.1. *Let assumptions* (H_b),(H_σ), (H'_ℓ) *and* ($H''_{\mathcal{K}}$) *be satisfied. Then* v_λ *is continuous in* $\mathcal{K}$.

Proof. Thanks to the invariance assumption ($H''_{\mathcal{K}}$), for any $x, x' \in \mathcal{K}$ one has $\mathcal{U}_{\mathcal{K}}(x) = \mathcal{U}_{\mathcal{K}}(x') = \mathcal{U}$ and the classical arguments usually applied to optimal control problems in the whole space can be used (see, for instance, [23, Proposition 2.1, Chapter II]). The proof is reported below for completeness.
Let us start observing that under assumption (H'_ℓ)

$$\mathbb{E}\left[\int_T^{+\infty} e^{-\lambda t}\ell(X_x^u(t), u(t))dt\right] \leq M_\ell \frac{1}{\lambda} e^{-\lambda T}, \qquad \forall x \in \mathcal{K}, u \in \mathcal{U}.$$

So we can choose T big enough such that $\mathbb{E}[\int_T^{+\infty} e^{-\lambda t}\ell(X_x^u(t), u(t))dt] < \varepsilon$. For any $\varepsilon > 0$ there exists a control $u_\varepsilon \in \mathcal{U}$ such that

$$\begin{aligned}
&v_\lambda(x) - v_\lambda(x') \\
&\leq J_\lambda(x, u_\varepsilon) - J_\lambda(x', u_\varepsilon) + \varepsilon \\
&\leq \mathbb{E}\left[\int_0^T e^{-\lambda t}\Big(\ell(X_x^u(t), u(t))dt - \ell(X_{x'}^u(t), u(t))dt\Big)\right] + 3\varepsilon \\
&\leq \mathbb{E}\left[\int_0^T e^{-\lambda t}\omega_\ell(|X_x^u(t) - X_{x'}^u(t)|)dt\right] + 3\varepsilon \\
&\leq \int_0^T e^{-\lambda t}\mathbb{E}\Big[\omega_\ell(|X_x^u(t) - X_{x'}^u(t)|)\Big]dt + 3\varepsilon \\
&\leq \int_0^T e^{-\lambda t}\omega_\ell\Big(\mathbb{E}\big[|X_x^u(t) - X_{x'}^u(t)|\big]\Big)dt + 3\varepsilon
\end{aligned}$$

where in the last passage we used the concavity of ω_λ for applying the Jensen inequality. Recalling that under assumptions (H_b) and (H_σ) the estimates of Proposition 2.1.1 in Chapter 2 hold, we have

$$v_\lambda(x) - v_\lambda(x') \leq L_\ell \int_0^T e^{-\lambda t}\omega_\ell\big(Ce^{CT}|x - x'|\big)dt + 3\varepsilon$$

where the constant C only depends on the Lipschitz constants of b and σ. Hence, choosing $|x - x'|$ small enough, we can conclude that $v_\lambda(x) - v_\lambda(x') < 4\varepsilon$. Reversing the role of x and x' and applying the same arguments we finally obtain the continuity of v_λ. □

Once proved its a priori continuity, v_λ can be characterized by standard dynamic programming techniques as a continuous solution, in viscosity sense, of the state constrained HJB equation

$$\begin{cases} \lambda v + H(x, Dv, D^2 v) = 0 & \text{in } int(\mathcal{K}) \\ \lambda v + H(x, Dv, D^2 v) \geq 0 & \text{in } \partial\mathcal{K} \end{cases} \tag{6.3.1}$$

where $int(\mathcal{K}) := \mathcal{K} \setminus \partial\mathcal{K}$ and $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}^d \to \mathbb{R}$ is defined by:

$$H(x, q, Q) := \sup_{u \in U}\Big\{ - q \cdot b(x, u) - \frac{1}{2}Tr[\sigma\sigma^T(x, u)Q] - \ell(x, u)\Big\}.$$

We point out that under the invariance assumption ($H''_{\mathcal{K}}$) for system (6.2.1) the dynamic programming techniques apply as in the unconstrained case and v_λ results to be also a sub-solution of (6.3.1) up to the boundary, as stated in the following:

Theorem 6.3.2. *Let assumptions* (H_b),(H_σ),(H'_ℓ) *and* $(H''_\mathcal{K})$ *be satisfied. Then* v_λ *is the unique bounded continuous viscosity solution to the HJB equation*

$$\lambda v + H(x, Dv, D^2v) = 0 \quad \text{in } \mathcal{K} \tag{6.3.2}$$

Proof. Thanks to the continuity of v_λ proved in Proposition 6.3.1, it is sufficient that $\mathcal{U}_\mathcal{K}(x) \neq 0$ for any $x \in \mathcal{K}$ for stating that v_λ satisfies a dynamic programming principle. Indeed, referring for instance to [99] or [55], it is possible to prove that for every stopping-time θ measurable with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ one has

$$v_\lambda(x) = \inf_{u\in\mathcal{U}_\mathcal{K}(x)} \mathbb{E}\left[v_\lambda(X_x^u(\theta)) + \int_0^\theta e^{-\lambda t}\ell(X_x^u(t), u(t))dt\right] \tag{6.3.3}$$

for every $x \in \mathcal{K}$.
The straightforward consequence of (6.3.3) is that v_λ is a viscosity solution of the state constrained HJB equation (6.3.1). Moreover we can prove that the sub-solution property holds also on $\partial\mathcal{K}$. In fact, since $\mathcal{U}_\mathcal{K}(x) = \mathcal{U}$ (consequence of $(H''_\mathcal{K})$-(ii)), any constant control $\nu(s) \equiv \nu \in U$ is admissible and by the DPP (6.3.3) we get

$$v_\lambda(x) \leq \mathbb{E}\left[v_\lambda(X_x^\nu(\theta)) + \int_0^\theta e^{-\lambda t}\ell(X_x^\nu(t), \nu(t))dt\right].$$

Given a test function φ such that $v_\lambda - \varphi$ has a strict maximum point in x such that $v_\lambda(x) = \varphi(x)$, for $\theta(\omega)$ small enough we have

$$\varphi(x) \leq \mathbb{E}\left[\varphi(X_x^\nu(\theta)) + \int_0^\theta e^{-\lambda t}\ell(X_x^\nu(t), \nu(t))dt\right].$$

Applying the Ito's formula and passing to the limit as θ goes to 0, we easily get

$$\lambda\varphi - D\varphi \cdot b(x,\nu) - \frac{1}{2}Tr[\sigma\sigma^T(x,\nu)D^2\varphi] - \ell(x,\nu) = 0$$

for any $\nu \in U$. Hence the sub-solution property is derived thanks to the arbitrariness if ν.
Finally, using the properties of H and the boundedness of the solutions, the uniqueness in the whole domain $\mathcal{K}$ follows by standard comparison arguments [110, Theorem 7.3], thanks to the fact that the equation is satisfied up to the boundary. ☐

6.4 Solution of the ergodic problem

In this section are contained the main results of the chapter obtained studying the asymptotic behavior of the problem for λ going to 0. The convergence of λv_λ is usually derived by an application of the Ascoli-Arzelà theorem once proved its equiboundedness and uniform equicontinuity. The following hypothesis is made in order to guarantee the equicontinuity peroperty:

(H_1) there exist two constants $C_1 \geq 0$ and $C_2 > 0$ such that

$$\mathbb{E}\left[|X_x^u(t) - X_y^u(t)|\right] \leq C_1 e^{-C_2 t}|x - y|$$

for any $u \in \mathcal{U}, x, y \in \mathcal{K}, t \geq 0$.

We will also consider the following finiteness assumption:

(H_2) for any compact set $\mathcal{C} \subseteq \mathcal{K}$ one has

$$\sup_{t\geq 0} \sup_{x\in\mathcal{C}} \sup_{u\in\mathcal{U}} \mathbb{E}\Big[|X_x^u(t)|\Big] < \infty$$

Property (H_1) was introduced in [38] and it is referred in literature with the name of *asymptotic flatness*. Let us define for $\varphi = b, \sigma$

$$\Delta_y\varphi(x,u) := \varphi(x+y,u) - \varphi(x,u)$$

and for $\varphi \in C^2(\mathbb{R}^d)$

$$\tilde{\mathcal{L}}^u\varphi(x;y) := \Delta_y b(x,u)D\varphi(y) + \frac{1}{2}Tr[\Delta_y\sigma\Delta_y\sigma^T(x,u)D^2\varphi(y)]$$

for any $x, y \in \mathbb{R}^d, u \in U$. A sufficient condition guaranteeing that (H_1) is satisfied is the existence of a C^2-Lyapunov function w such that there exist k_1, k_2, C_2 positive constants such that

$$\tilde{\mathcal{L}}^u w(x;y) \leq -C_2 w(y),$$

and

$$k_1|x| \leq w(x) \leq k_2|x|,$$

for any $u \in U$.
In particular, it is proved in [6] that if there exist a symmetric positive definite matrix $Q \in \mathbb{R}^{d\times d}$ and a constant $R > 0$ such that for every $u \in U$ and for every $x, y \in \mathbb{R}^d$ $(y \neq 0)$

$$2\Delta_y b(x,u)^T Qy - \frac{|\Delta_y\sigma(x,u)^T Qy|^2}{y^T Qy} + Tr\big[\Delta_y\sigma(x,u)\Delta_y\sigma(x,u)^T Q\big] \leq -R|y|^2 \tag{6.4.1}$$

then (H_1) holds and this Lyapunov function is given by

$$w(x) = (x^T Qx)^{\frac{1}{2}}.$$

By similar arguments can also be proved (see again [6]) that (6.4.1) implies also (H_2). Some examples for which (6.4.1) is satisfied are reported below.

Example 6.4.1. Let be $b(x,u) = Ax + Bu$ and $\sigma(x,u) = \{\sigma_i(x)\}_{i=1\ldots,p}$ with $\sigma_1(x) = x$ and $\sigma_i(x) = 0$ $(i \neq 1)$, where A and B are two constant matrices of the suitable size and σ_i denotes the i-th column vector of σ. If all the eigenvalues of A have negative real part has been proved in [38] and [6] that (6.4.1) is automatically satisfied.

Example 6.4.2. If a linear diffusion $\sigma(x,u) = Cx + Du$ is considered, for C, D matrices of suitable sizes, one has, taking for instance $Q = I$,

$$-\frac{|\Delta_y\sigma(x,u)^T y|^2}{|y|^2} + Tr\big[\Delta_y\sigma(x,u)\Delta_y\sigma(x,u)^T\big] = -\frac{|y^T C^T y|^2}{|y|^2} + |Cy|^2 \leq \|C\|^2|y|^2,$$

that means that condition (6.4.1) holds as soon as the drift b satisfies

$$2\Delta_y b(x,u) \leq (-R - \|C\|^2)|y|^2.$$

Thanks to the previous assumptions we can state the following result concerning equicontinuity estimates for the value function v_λ.

Proposition 6.4.1. *Let assumptions* (H_b),(H_σ),(H'_ℓ),$(H''_\mathcal{K})$ *and* (H_1) *be satisfied. Then there exists a constant* $C \geq 0$ *(independent of* λ*) such that*

$$|v_\lambda(x) - v_\lambda(x')| \leq \omega_\ell(C|x - x'|)$$

for any $x, x' \in \mathcal{K}$.

Proof. As in the proof of Proposition 6.3.1 we start observing that, thanks to assumption $(H''_\mathcal{K})$, one has $\mathcal{U}_\mathcal{K}(x) = \mathcal{U}_\mathcal{K}(x')$ for any $x, x' \in \mathcal{K}$. Therefore by (H'_ℓ) and the concavity of ω_ℓ the following inequalities hold:

$$\begin{aligned}
&|v_\lambda(x) - v_\lambda(x')| \\
&= |\inf_{u\in\mathcal{U}} J_\lambda(x, u) - \inf_{u\in\mathcal{U}} J_\lambda(x', u)| \\
&\leq \sup_{u\in\mathcal{U}} |J_\lambda(x, u) - J_\lambda(x', u)| \\
&\leq \sup_{u\in\mathcal{U}} \mathbb{E}\Big[\int_0^\infty e^{-\lambda t}|\ell(X_x^u(t)) - \ell(X_{x'}^u(t))|dt\Big] \\
&\leq \sup_{u\in\mathcal{U}} \int_0^\infty \mathbb{E}\Big[\omega_\ell(|X_x^u(t) - X_{x'}^u(t)|)\Big] \leq \sup_{u\in\mathcal{U}} \omega_\ell\Big(\int_0^\infty \mathbb{E}\big[|X_x^u(t) - X_{x'}^u(t)|\big]dt\Big).
\end{aligned}$$

Using now assumption (H_1) we obtain

$$|v_\lambda(x) - v_\lambda(x')| \leq \omega_\ell\big(|x - x'| \int_0^\infty C_1 e^{-C_2 t} dt\big)$$

and since $C_2 > 0$ the result follows by the convergence of the integral in last term. $\square$

We can now prove the following result:

Theorem 6.4.2. *Let assumptions* (H_b),(H_σ),(H'_ℓ),$(H''_\mathcal{K})$,(H_1) *be satisfied. Then* λv_λ *converges uniformly on* $\mathcal{K}$ *to a constant* Λ *as* λ *goes to* 0^+. *Moreover for any subsequence* λ_n *going to* 0^+, $u_{\lambda_n} - u_{\lambda_n}(0)$ *converges to a uniformly continuous function* χ *solution to the cell problem*

$$\Lambda + H(x, D\chi, D^2\chi) = 0 \qquad x \in \mathcal{K}. \tag{6.4.2}$$

Proof. Let us start by observing that there exists a constant C (independent of λ) such that for any $\lambda > 0$

$$|\lambda v_\lambda| \leq C \qquad \text{in } \mathcal{K}. \tag{6.4.3}$$

This can in fact be obtained either by the very definition of v_λ or as a consequence of the comparison principle for equation (6.3.2), taking into account sub-solutions (resp. super-solutions) of the form $-C/\lambda$ (resp. C/λ). The uniform bound (6.4.3) together with Proposition 6.4.1 allow us to apply the Ascoli-Arzelà theorem, then (extracting a subsequence if necessary) $\lambda v_\lambda - \lambda v_\lambda$ converges uniformly to 0. It follows that λv_λ converges uniformly to a constant in any compact set contained in $\mathcal{K}$ as λ goes to 0^+. Analogously $v_\lambda - v_\lambda(0)$ converges uniformly to a uniformly continuous function in any compact subset of $\mathcal{K}$. By a standard diagonal procedure we can finally extract a subsequence λ_n going to 0^+ such that $\lambda_n u_{\lambda_n}$ converges to a constant Λ and $u_{\lambda_n} - u_{\lambda_n}(0)$ to a uniformly continuous function χ in the whole domain $\mathcal{K}$. Since the convergence is locally uniform, by standard stability results for solution of HJB equations (see for instance [170, Theorem 6.8]), χ results a viscosity solution of (6.4.2). $\square$

We conclude the chapter with the following theorem that states that if the finiteness property (H_2) is satisfied, then Λ is actually the unique constant such that the cell problem admits a solution in the class of uniformly continuous functions.

Theorem 6.4.3. *If (H_2) holds, together with the assumptions of Theorem 6.4.2, the constant Λ given by Theorem 6.4.2 is the unique constant such that there exists a uniformly continuous solution to equation (6.4.2).*

Proof. Let us consider two pairs (Λ_1, χ_1) and (Λ_2, χ_2), where Λ_1, Λ_2 are two constant and χ_1, χ_2 two uniformly continuous functions on $\mathcal{K}$ such that for $i = 1, 2$

$$\Lambda_i + H(x, D\chi_i, D^2\chi_i) = 0 \qquad \mathcal{K}.$$

Let us also assume that $\Lambda_1 > \Lambda_2$ and let us define the function $v_i, i = 1, 2$ by

$$v_i(t, x) := \chi_i(x) + \Lambda_i t.$$

It is not difficult to prove that for $i = 1, 2$ respectively, v_i is a uniformly continuous viscosity solution to the parabolic equation

$$\begin{cases} \partial_t v + H(x, Dv, D^2v) = 0 & (0, +\infty) \times \mathcal{K} \\ v(0, x) = \chi_i(x) & \mathcal{K}. \end{cases} \tag{6.4.4}$$

Let us consider the following optimal control problem:

$$w_i(t, x) := \inf_{u \in \mathcal{U}_{\mathcal{K}}(x)} \mathbb{E}\left[\int_0^t \ell(X_x^u(s), u(s))ds + \chi_i(X_x^u(t))\right] \qquad i = 1, 2.$$

Thanks to our assumptions and the uniform continuity of the function χ_i, one can prove the uniform continuity of w_i for any $i = 1, 2$. Thanks to uniqueness results for viscosity solutions of (6.4.4) [91, Theorem 2.1] we can state that

$$v_i(t, x) = \inf_{u \in \mathcal{U}_{\mathcal{K}}(x)} \mathbb{E}\left[\int_0^t \ell(X_x^u(s), u(s))ds + \chi_i(X_x^u(t))\right] \qquad i = 1, 2.$$

For any $\varepsilon > 0$ let $u_\varepsilon \in \mathcal{U}_{\mathcal{K}}(x)$ be such that

$$v_2(t, x) \geq \mathbb{E}\left[\int_0^t \ell(X_x^{u_\varepsilon}(s), u_\varepsilon(s))ds + \chi_2(X_x^{u_\varepsilon}(t))\right] - \varepsilon, \tag{6.4.5}$$

then by the properties of the infimum one has

$$\begin{aligned} v_1(t, x) - v_2(t, x) &\leq \mathbb{E}\left[\int_0^t \ell(X_x^{u_\varepsilon}(s), u_\varepsilon(s))ds + \chi_1(X_x^{u_\varepsilon}(t))\right] + \\ &\quad -\mathbb{E}\left[\int_0^t \ell(X_x^{u_\varepsilon}(s), u_\varepsilon(s))ds + \chi_2(X_x^{u_\varepsilon}(t))\right] + \varepsilon \\ &\leq \mathbb{E}\left[\chi_1(X_x^{u_\varepsilon}(t)) - \chi_2(X_x^{u_\varepsilon}(t))\right] + \varepsilon \end{aligned}$$

and by the very definition of v_i, for $i = 1, 2$ it follows

$$\chi_1(x) - \chi_2(x) + (\Lambda_1 - \Lambda_2)t \leq \mathbb{E}\left[\chi_1(X_x^{u_\varepsilon}(t)) - \chi_2(X_x^{u_\varepsilon}(t))\right] + \varepsilon.$$

By the linear growth of χ_1 and χ_2 one obtains

$$\begin{aligned}\mathbb{E}\Big[\chi_1(X_x^{u_\varepsilon}(t)) - \chi_2(X_x^{u_\varepsilon}(t))\Big] &\leq \mathbb{E}\Big[|\chi_1(X_x^{u_\varepsilon}(t))| + |\chi_2(X_x^{u_\varepsilon}(t))|\Big] \\ &\leq C(1+\mathbb{E}\big[|X_x^{u_\varepsilon}(t))|\big])\end{aligned}$$

so

$$\chi_1(x) - \chi_2(x) + (\Lambda_1 - \Lambda_2 - \frac{\varepsilon}{t})t \leq C(1+\mathbb{E}\big[|X_x^{u_\varepsilon}(t))|\big]).$$

The last inequality yields a contradiction. In fact thanks to assumption (H_2) the right-hand side results is bounded for any $x \in \mathcal{C}$ (with $\mathcal{C}$ compact set in $\mathcal{K}$) and $t \geq 0$, so taking the limit for t that goes to $+\infty$ the contradiction follows since we assumed $\Lambda_1 > \Lambda_2$. It means that necessarily $\Lambda_1 \leq \Lambda_2$, but reversing the role of (Λ_1, χ_1) and (Λ_2, χ_2) the other inequality is obtained, that is finally $\Lambda_1 = \Lambda_2$. □

Conclusions and perspectives

The research developed in the present thesis provides a contribution to the study of different stochastic control problems involving state-constraints. We would like to mention below some of the future directions of research that follow, in our opinion, from this work.

In Chapter 3 we studied the characterization of backward reachability by a level set approach. In particular we asked the target and the state constraints to be satisfied almost surely in a finite horizon of time $T < +\infty$.
A future direction of research will be to extend this approach in order to take into account also the infinite horizon case $T = +\infty$ and be able to deal with different type of requirements as, for instance, the satisfaction of the target and/or the state constrained condition with a certain given probability $p \in (0, 1]$.
We also point out that the numerical examples proposed in Section 3.8 have the unique objective to prove the validity of our method and tests on application taken from concrete models have not yet been carried out.
Moreover the ideas developed in Section 3.7 for studying error estimates for semi-Lagrangian schemes for HJB equations with oblique derivative boundary conditions could be probably be exploited in other frameworks where the "shacking coefficient" techniques have to be adapted to the presence of a boundary.

In Chapter 4 we proposed a generalization of the Zubov method for characterizing the domain of asymptotic controllability for systems of controlled diffusions under state-constraints. Here we do not specify the exact probability of controllability. We conjecture that this approach can be extended to a characterization of the sets

$$\left\{ x \in \mathbb{R}^d : \sup_{u \in \mathcal{U}} \mathbb{P}\left[\lim_{t \to +\infty} d(X_x^u(t), \mathcal{T}) = 0 \text{ and } X_x^u(t) \in \mathcal{K}, \forall t \geq 0 \right] = p \right\}$$

for given probabilities $p \in [0, 1]$, similar to how [69] extends [73].

Chapter 5 concerns the study of state constrained optimal control. The approach we proposed is based on the formulation of the state constrained optimal control problem as a backward reachability problem which involves unbounded controls in an augmented state space. We solved such a problem by a level set approach obtaining a characterization of the level set function w, defined by (5.4.1), as the unique viscosity solution of the generalized HJB (5.5.2). The numerical approximation of this kind of equation has not been investigated in the thesis, but we can consider as a starting point the papers of Bokanowski, Brüder et al. [45] and Debrabant and Jakobsen [92]. In virtue of Theorem 5.4.3, solving numerically equation (5.5.2) would provide a numerical approximation of the value function v solution of the original state constrained optimal control problem

(5.2.3) and a further issue would be to compare our approach with other common techniques of treatment of the state-constraints (see for instance [39]).
Another direction of research is to further develop the compactification technique used in Section 5.5 (Lemma 5.5.4). In fact, our results seem to extend those in [145] and [144] taking into account the dependency of the diffusion on the unbounded control α. In our case such a dependence is linear since $\tilde{\sigma}(t, x, u, \alpha) := (\sigma^T(t, x, u), \alpha)^T$, but we think that more general cases could be taken into account.

In Chapter 6 we studied the ergodic problem associated with second order state constrained HJB equations. We presented a set of assumptions making the problem solvable. We don't have in mind for the moment how to replace the condition of invariance $(H''_{\mathcal{K}})$ on the set $\mathcal{K}$. By the way it seems that recently developed techniques (see [24]) might be employed in absence of the "asymptotic flatness" condition (H_1).

Bibliography

[1] M. Abu Hassan and C. Storey. Numerical determination of domains of attraction for electrical power systems using the method of Zubov. *Int. J. Control,*, 34:371–381, 1981.

[2] A. Altarovici, O. Bokanowski, and H. Zidani. A general Hamilton-Jacobi framework for non-linear state-constrained control problems. *ESAIM: COCV*, 19:337–357, 2013.

[3] O. Alvarez and M. Bardi. Ergodic problems in differential games. *Ann. Internat. Soc. Dynam. Games*, 9:131–152, 2007.

[4] O. Alvarez and M. Bardi. Ergodicity, stabilization, and singular perturbations for Bellman-Isaacs equations. *Mem. Amer. Math. Soc.*, 204(960), 2010.

[5] F. Antonelli. Backward-forward stochastic differential equations. *Ann. Appl. Probab.*, 3:777–793, 1993.

[6] A. Arapostathis, V.S. Borkar, and M. K. Ghosh. *Ergodic control of diffusion processes*. Cambridge University press, 2012.

[7] M. Arisawa. Ergodic problem for the Hamilton-Jacobi-Bellman equation I. *Ann. I.H.P. Anal. Non Lin.*, 14(4):415–438, 1997.

[8] M. Arisawa. Ergodic problem for the Hamilton-Jacobi-Bellman equation II. *Ann. I.H.P. Anal. Non Lin.*, 15(1):1–24, 1998.

[9] M. Arisawa and Lions P.L. On ergodic stochastic control. *Comm in Partial Differential Equations*, 23(11 and 12):2187–2217, 1998.

[10] Z. Artstein. Stabilization with relaxed controls. *Nonlinear Anal.*, 7:1163–1173, 1983.

[11] J.-P. Aubin. Viability Kernels and Capture Basins of sets under differential inclusions. *SIAM J. Control Optim.*, 40(3):853–881, 2001.

[12] J.-P. Aubin. Viability solutions to structured Hamilton-Jacobi equations under constraints. *SIAM J. Control Optim.*, 49:1881–1915, 2011.

[13] J.-P. Aubin and G. Da Prato. Stochastic viability and invariance. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 17(4):595–613, 1990.

[14] J.-P. Aubin and G. Da Prato. The viability theorem for stochastic differential inclusions. *Stochastic Anal. App.*, 16(1):1–15, 1998.

[15] J.-P. Aubin and H. Frankowska. *Set-Valued Analysis.* Birkhäuser Boston, 1990.

[16] J.-P. Aubin and H. Frankowska. The viability kernel algorithm for computing value functions of infinite horizon optimal control problems. *J. Math. Anal. App.*, 201:555–576, 1996.

[17] J.P. Aubin. *Viability Theory.* Birkhauser, Boston, 1991.

[18] J.P. Aubin and A. Cellina. *Differential Inclusions.* Springer, Berlin, 1984.

[19] J.P. Aubin and G. Da Prato. Stochastic Nagumo's viability theorem. *Stochastic Anal. Appl.*, 13:1–11, 1995.

[20] J.P. Aubin, G. Da Prato, and H. Frankowska. Stochastic invariance for differential inclusions. *Set-valued Analysis*, 8:181–201, 2000.

[21] J.P. Aubin and P. Saint-Pierre. *Viability theory. New directions.* 2011.

[22] B. Aulbach. Asymptotic stability regions via extensions of Zubov's method. I and II. *Nonlinear Anal., Theory Methods Appl.*, 7:1431–1440 and 1441–1454, 1983.

[23] M. Bardi and I. Capuzzo Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman equations.* Systems Control Found. Appl., Birkhäuser Boston, Inc., Boston, MA, 1997.

[24] M. Bardi, A. Cesaroni, and L. Rossi. A Liouville type result for a degenerate Bellman operator in a bounded domain. Preprint, 2014.

[25] M. Bardi and P. Goatin. Invariant sets for controlled degenerate diffusions: A viscosity so- lutions approach. *W. M. McEneaney, G. G. Yin and Q. Zhang (eds), Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W. H. Fleming*, pages 191–208.

[26] M. Bardi and R. Jensen. A Geometric Characterization of Viable Sets for Controlled Degenerate Diffusions. *Set-valued Analysis*, 10:129–141, 2002.

[27] G. Barles and J. Burdeau. The Dirichlet problem for semilinear second-order degenerate elliptic equations and applications to stochastic exit time control problems. *Commun in partial differential equations*, 20(1-2):129–178, 1995.

[28] G. Barles and F. Da Lio. On the boundary ergodic problem for fully nonlinear equations in bounded domains with general nonlinear Neumann boundary conditions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22(5):521–541, 2005.

[29] G. Barles, C. Daher, and M. Romano. Optimal control on the l^∞ norm of a diffusion process. *SIAM J. Control Optim.*, 32(3):612–634, 1994.

[30] G. Barles, C. Daher, and M. Romano. Convergence of numerical schemes for parabolic equations arising in finance theory. *M3AS*, 5(1):125–143, 1995.

[31] G. Barles and E.R. Jakobsen. On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations. *M2AN Math. Model. Numer. Anal.*, 36:33–54, 2002.

[32] G. Barles and E.R. Jakobsen. Error bounds for monotone approximation schemes for Hamilton-Jacobi-Bellman equations. *SIAM J. Numer. Anal.*, 43(2):540–558, 2005.

[33] G. Barles and E.R. Jakobsen. Error bounds for monotone approximation schemes for parabolic Hamilton-Jacobi-Bellman equations. *Mathematics of Computation*, 74(260):1861–1893, 2007.

[34] G. Barles and Lions P.L. Fully nonlinear Neumann type boundary conditions for first-order Hamilton-Jacobi equations. *Nonlinear Anal. Theory Methods Appl.*, 16(2):143–153, 1991.

[35] G. Barles and E. Rouy. A strong comparison result for the bellman equation arising in stochastic exit time control problems and its applications. *Commun. in partial differential equations*, 23(11–12):1945–2033, 1998.

[36] G. Barles and P.E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Anal.*, 4:271–283, 1991.

[37] E. N. Barron. The Bellman equation for control of the running max of a diffusion and applications to lookback options. *Applicable Analysis*, 48:205–222, 1993.

[38] G.K. Basak, V.S. Borkar, and M.K. Gosh. Ergodic control of degenerate diffusions. *Stochastic Anal. Appl.*, 15(1):1–17, 1997.

[39] M. Basei, A. Cesaroni, and T. Vargiolu. Optimal exercise of swing contracts in energy markets: an integral constrained stochastic optimal control problem. *SIAM J. Finan. Math.*, 5(1):581–608, 2014.

[40] R. Bellman. *Dynamic Programming.* Princeton University Press, 1957.

[41] Benes. Existence of optimal stochastic controls. *SIAM J. Control*, 9:446–472, 1971.

[42] A. Bensoussan. On Bellman equations of ergodic control in $\mathbb{R}^n$. *J reine angew. Math.*, 429:125–160, 1992.

[43] A. Bensoussan and J. Frehese. Ergodic control bellman equation with neumann boundary conditions. *Lecture Notes in Control and Inform. Sci.*, 280:59–71, 2002.

[44] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–659, 1973.

[45] O. Bokanowski, B. Brüder, S. Maroso, and H. Zidani. Numerical approximation for a superreplication problem under gamma contraints. *SIAM J. of Numer. Analysis*, 47(3):2289–2320, 2009.

[46] O. Bokanowski, Y. Cheng, and C.W. Shu. A discontinuous Galerkin scheme for front propagation with obstacles. *Numerische Mathematik*, 126:1–31, 2014.

[47] O. Bokanowski, N. Forcadel, and H. Zidani. Reachability and minimal times for state constrained nonlinear problems without any controllability assumption. *SIAM J. Control Optim.*, 48(7):4292–4316, 2010.

[48] J.F. Bonnans, S. Maroso, and H. Zidani. Error bounds for stochastic differential games: the adverse stopping case. *IMA, J. Numer. Anal.*, 26(1):188–212, 2006.

[49] J.F. Bonnans, S. Maroso, and H. Zidani. Error bounds for a Stochastic impulse Control Problem. *Appl. Math Optim. (AMO)*, 55(3):327–357, 2007.

[50] J.F. Bonnans, E. Ottenwaelter, and H. Zidani. Numerical schemes for the two dimensional second-order HJB equation. *ENSAIM: M2AN*, 38:723–735, 2004.

[51] J.F. Bonnans and H. Zidani. Consistency of generalized finite difference schemes for the stochastic HJB equation. *SIAM J. Numer. Anal.*, 41(3):1008–1021, 2003.

[52] V. Borkar and A. Budhiraja. Ergodic control for constrained diffusions: characterization using HJB equations. *SIAM J. Control Optim.*, 43(4):1467–1492, 2005.

[53] V. Borkar and M. K. Ghosh. Ergodic control of Multidimensional Ddffusions I: The Ei:stence Rreslts. *SIAM J. Control Optim.*, 26(1):112–126, 1988.

[54] V. Borkar and M. K. Ghosh. Ergodic control of Multidimensional Diffusions: II Adaptive Control. *Appl. Math Optim. (AMO)*, 21:191–220, 1990.

[55] V.S. Borkar. *Optimal control of Diffusion Processes.* Longman Scientific and Technical, Harlow, UK, 1989.

[56] B. Bouchard. Stochastic Targets with mixed diffusion processes and viscosity solutions. *Stochastic Processes and their Applications*, 101:273–302, 2002.

[57] B. Bouchard. Optimal reflection of diffusions and barrier options pricing under constraints. *SIAM J. Control and Optim.*, 47(4):1785–1813, 2008.

[58] B. Bouchard and N. M. Dang. Optimal Control versus Stochastic Target problems: an equivalence result. *Systems and Control Letters*, 61(2):343–346, 2012.

[59] B. Bouchard and N. M. Dang. Generalized stochastic target problems for pricing and partial hedging under loss constraints–application in optimal book liquidation. *Finance & Stochastics*, 17(1):31–72, 2013.

[60] B. Bouchard, R. Elie, and C. Imbert. Optimal control under stochastic target constraints. *SIAM J. Control Optim.*, 48(5):3501–3531, 2010.

[61] B. Bouchard, R. Elie, and N. Touzi. Stochastic target problems with controlled loss. *SIAM J. Control Optim.*, 48(5):3123–3150, 2009.

[62] B. Bouchard and M. Nutz. Weak dynamic programming for generalized state constraints. *SIAM J. Control Optim.*, 50(6):3344–3373, 2012.

[63] B. Bouchard and N. Touzi. Weak dynamic programming principle for viscosity solutions. *SIAM J. Control Optim.*, 49(3):948–962, 2011.

[64] B. Brüder. Super-replication of European options with a derivative asset under constrained finite variation strategies. Preprint HAL http://hal.archives-ouvertes.fr/hal-00012183/fr/, 2005.

[65] R. Buckdahn, S. Peng, M. Quincampoix, and C. Rainer. Existence of stochastic control under state constraints. *C.R. Acad. Sci. Paris Ser. I Math.*, 327:17–22, 1998.

[66] A. Budhiraja. An Ergodic control problem for constrained diffusion processes: existence of optimal Markov control. *SIAM J. Control Optim.*, 42(2):532–558, 2001.

[67] F. Camilli, A. Cesaroni, L. Grüne, and F. Wirth. Stabilization of controlled diffusions via Zubov's method. *Stochastics & Dynamics*, 6:373–394, 2006.

[68] F. Camilli and M. Falcone. An approximation scheme for the optimal control of diffusion processes. *RAIRO Modél. Math. Anal. Numér.*, 29(1):97–122, 1995.

[69] F. Camilli and L. Grüne. Characterizing attraction probabilities via the stochastic zubov method. *Discrete Contin. Dynam. Systems B*, 3:457–468, 2003.

[70] F. Camilli, L. Grüne, and F. Wirth. A generalization of the Zubov's equation to perturbed systems. *SIAM J. Control Optim.*, 40:496–515, 2002.

[71] F. Camilli, L. Grüne, and F. Wirth. Characterizing controllability probabilities of stochastic control systems via Zubov's method. In *Proceedings of the 16th International Symposium on Mathematical Theory of Networks and Systems (MTNS2004)*, 2004.

[72] F. Camilli, L. Grüne, and F. Wirth. Control Lyapunov Functions and Zubov's Method. *SIAM J. Control Optim.*, 47:301–326, 2008.

[73] F. Camilli and P. Loreti. A characterization of the domain of attraction for a locally exponentially stable stochastic system. *NoDea*, 13:205–222, 2006.

[74] I. Capuzzo Dolcetta. On a discrete approximation of the hamilton-jacobi equation of dynamic programming. *Appl. Math. Optim.*, 10:367–377, 1983.

[75] I. Capuzzo Dolcetta and Garroni. Oblique derivative problems and invariant measures. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 13(4):689–720, 1986.

[76] I. Capuzzo Dolcetta and J. Menaldi. On the deterministic optimal stopping time problem in the ergodic case. In *Theory and Applications for Nonlinear Control Systems, North-Holland, Amsterdam*, pages 453–460, 1986.

[77] I. Capuzzo Dolcetta and Lions P.L. Hamilton-jacobi equations with state constraints. *Trans. Amer. Math. Soc.*, 318:643–683, 1990.

[78] P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre. Optimal times for constrained nonlinear control problems without local contrllability. *Appl. Math. Optim.*, 36:21–42, 1997.

[79] P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre. Numerical schemes for discontinuous value function of optimal control. *Set-valued Analysis*, 8:111–126, 2000.

[80] A. Cesaroni. Lyapunov stabilizability of controlled diffusions via a superoptimality principle for viscosity solutions. *Appl. Math. Optim.*, 53(1):1–29, 2006.

[81] F. Clarke. *Functional Analysis, calculus of variations and optimal control.* Graduate Texts in Mathematics, Springer, 2013 edition.

[82] F. Clarke, Yu.S. Ledyaev, L. Rifford, and R. J. Stern. Feedback Stabilization and Lyapunov Functions. *SIAM J. Control Optim.*, 39(1):25–48, 2000.

[83] M.G. Crandall and H. Ishii. The maximum principle for semicontinuous functions. *Differential Integral Equations*, 3:1001–1014, 1990.

[84] M.G. Crandall, H. Ishii, and P.L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, 27(1):1–67, 1992.

[85] M.G. Crandall and P.L. Lions. Condition d'unicité pour les solutions generalisées des équations de Hamilton-Jacobi du premier ordre. *C. R. Acad. Sci. Paris Sér. I Math.*, 292:183–186, 1981.

[86] M.G. Crandall and P.L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277:1–42, 1983.

[87] M.G. Crandall and P.L. Lions. Convergent difference schemes for nonlinear parabolic equations and mean curvature motion. *Math. Comp.*, 75(1):17–41, 1996.

[88] J. Cvitanić and I. Karatzas. Hedging contingent claims with constrained portfolios. *Ann. Appl. Probab.*, 3:652–681, 1993.

[89] J. Cvitanić and J. Ma. Hedging options for a large investor and forward-backward SDE's. *Ann. Appl. Probab.*, 6:370–398, 1996.

[90] J. Cvitanic, H. Pham, and N. Touzi. Super-replication in stochastic volatility models with portfolio constraints. *J. Appl. Probab.*, 36:523–545, 1999.

[91] F. Da Lio and O. Ley. Uniqueness Results for Second Order Bellman-Isaacs Equations under Quadratic Growth Assumptions and Applications. *SIAM J. Control Optim.*, 45(1):74–106, 2006.

[92] K. Debrabant and E.R. Jakobsen. Semi-Lagrangian schemes for linear and fully non-linear diffusion equations. *Math. Comp.*, 82(283):1433–1462, 2012.

[93] P. Dupuis and H. Ishii. On oblique derivative problems for fully nonlinear second-order elliptic partial differential equations on nonsmooth domains. *Nonlinear Anal. Theory Methods Appl.*, 15(12):1123–1138, 1990.

[94] N. El Karui, D. H. Nguyen, and M. Jeanblanc-Picqué. Compactification methods in the control of degenerate diffusions: Existence of an optimal control. *Stochastics*, 20:169–219, 1987.

[95] N. El Karui and M.-C. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.*, 33:29–66, 1995.

[96] L. C. Evans. On solving certain nonlinear differential equations by accretive operator methods. *Israel J. Math.*, 36:225–247, 1980.

[97] L.C. Evans. A convergence theorem for solutions of nonlineat second order elliptic equations. *Indiana Univ. Math. J.*, 27:875–887, 1978.

[98] M. Falcone, T. Giorgi, and P. Loreti. Level sets of viscosity solutions: Some applications to fronts and rendez-vous problems. *SIAM J. Appl. Math.*, 54:1335–1354, 1994.

[99] W. H. Fleming and H. M. Soner. *Controlled Markov Processes and Viscosity Solutions, 2nd ed.* Springer, 2006.

[100] H. Frankowska and M. Mazzola. Discontinuous solutions of Hamilton-Jacobi-Bellman equation under state constraints. *Calculus of Variations and Partial Differential Equations*, 46(3-4):725–747, 2013.

[101] H. Frankowska and R. B. Vinter. Existence of Neighboring Feasible Trajectories: Applications to Dynamic Programming for state-constrained optimal control problems. *Journal of Optimization Theory and Applications*, 104(1):20–40, 2000.

[102] A. Friedman. *Stochastic Differential Equations and Applications*, volume 2. Academic Press, New York, 1976.

[103] R. Genesio, M. Tartaglia, and A. Vicino. On the estimation of asymptotic stability regions: State of the art and new proposals. *IEEE Trans. Autom. Control*, 30:747–755, 1985.

[104] L. Grüne and H. Zidani. Zubov's equation for state-constrained perturbed nonlinear systems. *Math. Control Related Fields*, 2014. to appear.

[105] W. Hahn. *Stability of motion.* 1967. Translated from the German manuscript by Arne P. Baartz. Die Grundlehren der mathematischen Wissenschaften, Band 138.

[106] R.Z. Hasminskii. *Stochastic stability of differential equations.* Sjithoff and Noordhoff Inter-national Publishers, 1980.

[107] U. G. Haussmann and J. P. Lepeltier. On the existence of optimal controls. *SIAM J. Control Optim.*, 28(4):851–902, 1990.

[108] A. C. Heinricher and R. H. Stockbridge. Optimal control of the running max. *SIAM J. Control Optim.*, 29(4):936–953, 1991.

[109] H. Ishii. A boundary value problem of the dirichlet type for Hamilton-Jacobi equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 16(4):105–135, 1989.

[110] H. Ishii. On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDEs. *Comm. Pure Appl. Math.*, 42:14–45, 1989.

[111] H. Ishii. Fully nonlinear oblique derivative problems for nonlinear Second-order elliptic PDEs. *Duke Math. J.*, 62(3):633–661, 1991.

[112] H. Ishii and S. Koike. A new formulation of state constraint problems for first-order PDE's. *SIAM J. Control Optim.*, 34(2):554–571, 1996.

[113] H. Ishii and P. L. Lions. Viscosity solutions of fully nonlinear Second-order elliptic partial differential equations. *J. Differential equations*, 83:26–78, 1990.

[114] H Ishii and P. Loreti. A class of stochastic optimal control problems with state constraint. *Indiana Univ. Math. J.*, 51(5):1167–1196, 2002.

[115] R. Jensen. The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. *Archive for Rational Mechanics and Analysis*, 101(1):1–27, 1988.

[116] R. Jensen. Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations. *Indiana Univ. Math. J.*, 38:929–667, 1989.

[117] M. A. Katsoulakis. Viscosity solutions of second fully nonlinear elliptic equations with state constraints. *Indiana Univ. Math. J.*, 43(2):493–519, 1994.

[118] N.E. Kirin, R. A. Nelepin, and V.N. Bajdaev. Construction of the attraction region by Zubov's method. *Differ. Equations*, 17:871–880, 1982.

[119] N. V. Krylov. On the rate of convergence of finite-difference approximations for Bellman's equations. *St. Petersburg Math. J.*, 9:639–650, 1997.

[120] N. V. Krylov. On the rate of convergence of finite-difference approximations for bellman's equations with variable coefficients. *Probab. Theory Relat. Fields*, 117:1–16, 2000.

[121] A. B. Kurzhanski, I. M. Mitchell, and P. Varaiya. Optimization techniques for state-constrained control and obstacle problems. *J. Optim. Theory Appl.*, 128:499–521, 2006.

[122] A. B. Kurzhanski and P. Varaiya. Ellipsoidal techniques for reachability under state constraints. *SIAM J. Control Optim.*, 45:1369–1394, 2006.

[123] Y. Kurzweil. On the inversion of the second theorem of Lyapunov on stability of motion. *Czechoslovak Math. J.*, 81(6):217–259, 455–473, 1956.

[124] H. Kushner. Converse theorems for stochastic Lyapunov functions. *SIAM J. Control Optim.*, 5:228–233, 1967.

[125] H. Kushner. *Stochastic stability and control.* 1967.

[126] H. Kushner. Stochastic stability. In *"Stability of stochastic dynamical systems" (Proc. In- ternat. Sympos., Univ. Warwick, Coventry, 1972)*, volume 294. Lecture Notes in Math., Springer, Berlin, 1972.

[127] H.J. Kushner. Existence results for optimal stochastic controls. *Journal of Optimization Theory and Applications*, 15(4):347–359, 1975.

[128] H.J. Kushner. *Probability Methods for Approximations in Stochastic Control and for Elliptic Equations.* Academic Press, New York, 1977.

[129] H.J. Kushner and P. Dupuis. *Numerical methods for stochastic control problems in continuous time.* Springer Verlag, New York.

[130] J.M. Lasry. *Contrôle stochastique ergodique.* PhD thesis, University of Paris IX, 1974.

[131] J.M. Lasry and P.L. Lions. Nonlinear Elliptic Equations with Singular Boundary Conditions and Stochastic Control with State Constraints. I. The Model Problem. *Math. Ann.*, 283(4):583–630, 1989.

[132] P.L. Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. Part I: The Dynamic Programming principle and applications. *Comm in Partial Differential Equations*, 8(10):1101–1174, 1983.

[133] P.L. Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. Part II: Viscosity solutions and uniqueness. *Comm in Partial Differential Equations*, 8(11):1229–1276, 1983.

[134] P.L. Lions. Neumann type boundary conditions for Hamilton-Jacobi equations. *Duke Math J.*, 52(4):793–820, 1985.

[135] P.D. Loewen. Existence Theory for a Stochastic Bolza Problem. *IMA J. Mathematical Control and Information*, 4:301–320, 1987.

[136] P. Loreti. Some properties of constrained viscosity solutions of Hamilton-Jacobi-Bellman equations. *SIAM J. Control Optim.*, 25:1244–1252, 1987.

[137] A. M. Lyapunov. *The General Problem of the Stability of Motion (English transl.)*. 1992.

[138] J. Lygeros. On reachability and minimum cost optimal control. *Automatica*, 40:917–927, 2004.

[139] J. Lygeros, C. Tomlin, and S. Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, 35:349–370, 1999.

[140] J.L. Massera. On Lyapunov's condition of stability. *Annals of Math.*, 50:705–721, 1949.

[141] J.L. Menaldi. Some estimates for finite difference approximations. *SIAM J. Control Optim.*, 27:579–607, 1989.

[142] R.C. Merton. Optimal consumption and portfolio rules in continuous time. *J. Economic Theory*, 3:373–413, 1971.

[143] M. Motta. On nonlinear optimal control problems with state constraints. *SIAM J. Control Optim.*, 33:1411–1424, 1995.

[144] M. Motta and C. Sartori. Uniqueness results for boundary value problems arising from finite fuel and other singular and unbounded stochastic control problems. *Discrete Contin. Dynam. Systems*, 21(2):513–535, 2008.

[145] M. Motta and C. Sartori. Weakly Coercive Problems in Nonlinear Stochastic Control: Existence of Optimal Controls. *SIAM J. Control Optim.*, 48(5):3532–3561, 2010.

[146] N. Nagumo. Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen. *Proc. Phys. Math. Soc. Japan*, 24, 1942.

[147] S. Osher and A.J. Sethian. Fronts propagating with curvature dependent speed: algorithms on Hamilton-Jacobi formulations. *J. Comp. Phys.*, 79:12–49, 1988.

[148] E. Pardoux. Backward stochastic differential equations and viscosity solutions of semilinear parabolic PDE's of second order. *Proceedings of Geilo Conference, Geilo, Norway*, 1996.

[149] E. Pardoux and S. Tang. Forward-backward stochastic differential equations and quasilinear parabolic PDE's. *Probab. Theory Related Fields*, 114:123–150, 1999.

[150] H. Pham. On some recent aspects of stochastic control and their applications. *Probability Surveys*, 2:506–549, 2005.

[151] Lions P.L. and B. Perthame. Quasi-variational inequalities and ergodic impulse control. *SIAM J. Control Optim.*, 24:604–615, 1986.

[152] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkriledze, and E.F. Mischenko. *The Mathematical Theory of Optimal Processes*. 1962.

[153] F. Rampazzo and C. Sartori. The Minimum Time Function with Unbounded Controls. *J. Math. Systems*, 8(2):1–34, 1998.

[154] F. Rampazzo and C. Sartori. Hamilton-Jacobi-Bellman equations with fast gradient-dependence. *Indiana Univ. Math. J.*, 49(3):1043–1077, 2000.

[155] L. Rifford. Semiconcave Control-Lyapunov Functions and Stabilizing Feedbacks. *SIAM J. Control Optim.*, 41(3):659–681, 2002.

[156] M. Robin. On some impulse control problems with ion run average control. *SIAM J. Control Optim.*, 19:333–358, 1981.

[157] M. Robin. Long-term average cost control problems for Continuous time Markov processes: A Survay. *Acta Applicandae Mathematicae*, 1:281–299, 1983.

[158] P. Saint-Pierre. Approximation of the viability kernel. *Appl. Math Optim. (AMO)*, 29:187–209, 1994.

[159] H. Soner and N. Touzi. Super-replication under Gamma constraints. *SIAM J.Control Optim.*, 39:73–96, 2000.

[160] H.M. Soner. Optimal control with state-space constraint. i. *SIAM J. Control Opt.*, 24(3):552–561, 1986.

[161] H.M. Soner. Optimal control with state-space constraint. ii. *SIAM J. Control Opt.*, 24(6):1110–1122, 1986.

[162] H.M. Soner and N. Touzi. Dynamic programming for stochastic target problems and geometric flows. *J. Eur. Math. Soc.*, 4:201–236, 2002.

[163] H.M. Soner and N. Touzi. Stochastic target problems, dynamic programming and viscosity solutions. *SIAM J. Control Opt.*, 41(2):404–424, 2002.

[164] E. Sontag. A Lyapunov-like characterization of asymptotic controllability. *SIAM J. Control Opt.*, 21:462–471, 1983.

[165] E. Sontag and H.J. Sussmann. Nonsmooth control-Lyapunov functions. In *Decision and Control, Proceedings of the 34th IEEE Conference*, volume 3, 1995.

[166] P. Soravia. Optimality principles and representation formulas for viscosity solutions of Hamilton-Jacobi equations. ii. Equations of control problems with state constraints. *Differential Integral Equations*, 12(2):275–293, 1999.

[167] P. Soravia. Optimality principles and representation formulas for viscosity solutionsof Hamilton-Jacobi equations. I. Equations of unbounded and degenerate control problems without uniqueness. *Adv. Differential Equations*, 4(2):275–296, 1999.

[168] R. J. Stern. Characterization of the state constrained minimal time function. *SIAM J. Control Optim.*, 43(2):697–707, 2004.

[169] R. Tarres. Comportement asymptotique d'un problème de contrô1e stochastique. *Cahier de Math. de la décision*, 1982. No. 8215, Université de Paris IX.

[170] N. Touzi. *Optimal Stochastic Control, Stochastic Target Problems, and Backward SDE.* Fields Institute Monographs, vol.49, Springer, 2012.

[171] P. Wilmott. *Paul Wilmott on quantitative finance.* John Wiley & Sons Ltd, England (Revised Ed. of).

[172] J. Yong and X.Y. Zhou. *Stochastic controls: Hamiltonian systems and HJB equations.* Stochastic Modelling and Applied Probability, vol.43, Springer.

[173] T. Yoshizawa. *Stability theory by Lyapunov's second method.* Mathematical Society of Japan, Tokyo, 1966.

[174] V.I. Zubov. Methods of A.M. Lyapunov and their Application. *P. Noordhoff, Groningen*, 1964.

Abstract. This thesis deals with Hamilton-Jacobi-Bellman (HJB) approach for some stochastic control problems in presence of state constraints. This class of problems arises in many challenging applications, and a wide literature has already analysed such problems under some strong controllability conditions. The main feature of the present thesis is to provide new ways to face the presence of constraints without assuming any controllability condition. The first important contribution in this direction is obtained by exploiting the existing link between backward reachability and optimal control. It is shown that by considering a suitable auxiliary unconstrained optimal control problem, the level set approach can be extended to characterize the backward reachable sets under state constraints. On the other hand the value function associated with a general state constrained stochastic optimal control problem is characterized by means of a state constrained backward reachable set, enabling the application of the level set method for handling the presence of the state constraints. This link between optimal control problems and reachability led to the theoretical and numerical analysis of HJB equations with oblique derivative boundary conditions and problems with unbounded controls. Error estimates for Markov-chain approximation represent another contribution of this manuscript. Furthermore, the properties of asymptotic controllability of a stochastic system have also been studied and a generalization of the Zubov method to state constrained stochastic systems is presented. In the last part of the thesis an ergodic optimal control problems in presence of state-constraints are considered.

Résumé. Cette thèse concerne l'approche Hamilton-Jacobi-Bellman (HJB) pour des problèmes de contrôle stochastique en présence de contraintes sur l'état du système. Cette classe de problèmes se pose dans de nombreuses applications importantes, et une grande littérature les a déjà analysé sous des conditions de contrôlabilité fortes. La principale contribution de cette thèse est de fournir de nouvelles façons de affronter la présence de contraintes sans hypothèse de contrôlabilité. Une première importante contribution dans cette direction est obtenue en exploitant le lien existant entre l'atteignabilité des systèmes stochastiques et des problèmes de contrôle optimal. Il est montré que, en considérant un problème approprié auxiliaire de la commande optimale sans contraintes sur l'état, l'approche level-set peut être étendue pour caractériser les ensembles atteignables sous contrainte sur l'état. D'autre part l'épigraphe de la fonction valeur associée à un problème général de commande optimale stochastique sous contraintes d'état peut être caractérisée par un ensemble atteignable d'un système dynamique augmenté. Ce résultat permet l'application de la méthode level-set pour gérer la présence des contraintes sur l'état sans faire d'hypothèse de contrôlabilité. Ce lien entre les problémes de contrôle optimal et les level-set a conduit à l'analyse théorique et numérique des équations HJB avec conditions aux limites de derivé obliques et de problèmes avec contrôles non bornés. Les estimations d'erreur d'approximation de type Chaine de Markov représentent une autre contribution de ce manuscrit. En outre, les propriétés de contrôlabilité asymptotique d'un système stochastique ont également été analysées et une généralisation de la méthode de Zubov aux systèmes stochastiques contraints est étudiée dans le manuscrit. La dernière partie de la thèse est dédié à l'étude de problèmes de contrôle optimal ergodiques en présence de contraintes sur l'état.